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THE

PHYSICAL REVIEW.

VISCOSITY AND FLUIDITY[1]—A SUMMARY OF RESULTS.[2] I.

By Eugene C. Bingham.

I. Theoretical Relation Between Viscosity and Fluidity.

THE coefficient of fluidity is measured by the velocity given to either of two horizontal planes, at unit distance apart, in respect to the other, by a unit tangential force per unit area, the space between the planes being filled with the viscous substance. But it is also the reciprocal of the well-known coefficient of viscosity, *i. e.*, $\varphi = 1/\eta$. Vis-

[1] Fifteenth communication bearing upon this subject.

[2] 1. Bingham, The Conductivity and Viscosity of Solutions of Certain Salts in Mixtures of Acetone with Methyl Alcohol, with Ethyl Alcohol, and Water. Dissertation, Johns Hopkins University, 1905.

2. Jones and Bingham, do., Amer. Chem. J., *34*, 481 (1905).

3. Bingham, Viscosity and Fluidity—Liquid Mixtures. Amer. Chem. J., *35*, 195 (1906).

4. Jones, Bingham, and McMaster, "Über Leitfähigkeit und innere Reibung von Lösungen gewisser Salze in den Lösungsmittelgemischen: Wasser, Methylalkohol, Äthylalkohol, und Aceton," Zeitschr. f. physik. Chem., *57*, 193 (1906).

5. Jones, Lindsay, Carroll, Bassett, Bingham, Rouiller, McMaster, and Veazey, Conductivity and Viscosity in Mixed Solvents. Carnegie Institution of Washington Publication No. *80* (1907).

6. Bingham, "Viscosity and Fluidity" (Preliminary Paper), Amer. Chem. J., *40*, 277 (1908).

7. Bingham, "Algebraische Analyse der Viskositätsdaten," Zeitschr. f. physik. Chem. *66*, 238 (1909).

8. Bingham and Miss Harrison, "Viskosität und Fluidität," Zeitschr. f. physik. Chem., *66*, 1 (1909).

9. Bingham, "Viscosity and Fluidity," Amer. Chem. J., *43*, 287 (1910).

10. Bingham, "Viscosity and Fluidity of Matter in the Three States of Aggregation and the Molecular Weight of Solids," Amer. Chem. J., *45*, 264 (1911).

11. Bingham and White, "The Viscosity and Fluidity of Emulsions, Crystalline Liquids and Colloidal Solutions," J. Amer. Chem. Soc., *33*, 1257 (1911).

12. Bingham and Durham, "The Viscosity and Fluidity of Suspensions of Finely-Divided Solids in Liquids," Amer. Chem. J., *46*, 278 (1911).

13. Bingham, "Fluidity and Vapor Pressure," Amer. Chem. J., *47*, 185 (1912).

14. Bingham and White, "Fluidität und die Hydrattheorie. I. Die Viskosität von Wasser," Zeitschr. f. physik. Chem., *80*, 670 (1912).

cosity and fluidity are analogous to electrical resistance and conductance respectively, the Law of Poiseuille,

$$\frac{8l}{\pi r^4 t} \cdot v = \frac{p}{\eta}$$

being the analogue of Ohm's Law,

$$C = \frac{E}{R}$$

since $8l/\pi r^4 t$ may be made constant, where v is the velocity of transpiration, p the pressure, η the viscosity, C the current, E the electromotive force, and R the resistance.

Case I. Fluidities are Additive.—Let Fig. 1 represent a section through a series of parallel lamellæ. If the surface AB is maintained at a constant difference of potential above CD, the total current C which will flow will be the sum of the partial currents flowing in the different lamellæ. Thus

$$C = [c_a + c_b + \cdots] = \frac{E}{r_a} + \frac{E}{r_b} + \cdots = \frac{E}{R},$$

where r_a, r_b, $\cdots$ represent the resistances of their respective lamellæ a, b, $\cdots$ and R the total resistance between AC and BD. So if AC is sheared downward with respect to BD, the resulting viscous flow Q is the sum of the amounts of flow in each lamella.

$$Q = [q_a + q_b + \cdots] = \frac{p}{\eta_a} + \frac{p}{\eta_b} + \cdots = \frac{p}{H},$$

where η_a, η_b, $\cdots$ represent the viscous resistances of their respective lamellæ and H the total viscous resistance between AC and BD. Therefore

$$\frac{1}{H} = \frac{1}{\eta_a} + \frac{1}{\eta_b} + \cdots$$

or

$$\Phi = \varphi_a + \varphi_b + \cdots$$

and the coefficient of fluidity will be

$$\varphi = m_1\varphi_1 + m_2\varphi_2 + \cdots, \qquad (1)$$

where m_1, m_2, $\cdots$ represent the fractions of the total volume occupied by the respective components present whose coefficients of fluidity are φ_1, φ_2, $\cdots$. As a convention we will suppose the components to be arranged in the order of increasing fluidities.

Case II. Viscosities are Additive.[1]—If the surface CA is at a constant difference of potential above BD, the resistance to the flow of the current will be the sum of the resistances of the different lamellæ

$$C = \frac{E}{r_a + r_b + \cdots} = \frac{E}{R}.$$

Or if we imagine the lamellæ capable of viscous flow and that the surface AB is sheared toward the right, the resistance to flow will be the sum of the separate resistances

$$Q = \frac{p}{\eta_a + \eta_b + \cdots} = \frac{p}{H'},$$

and therefore

$$H' = \eta_a + \eta_b + \cdots$$

and the coefficient of viscosity will be

$$\eta' = m_1\eta_1 + m_2\eta_2 + \cdots. \qquad (2)$$

The analogy is so nearly perfect that it is necessary to guard against one point of difference. In electrical flow, the conductors themselves remain stationary, but in viscous flow the substances themselves are

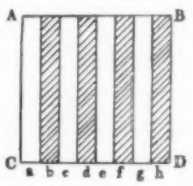


Fig. 1.

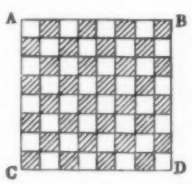


Fig. 2.

deformed. If the two above schemes are combined, a checkerboard arrangement (Fig. 2) is obtained and the electrical resistance of such a system of conductors is intermediate between those of the simple patterns. Thus one might be led to suppose that in mixtures of liquids, the viscous

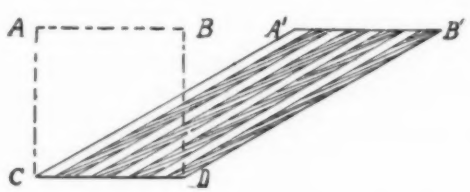


Fig. 3.

[1] Cp. eleventh paper.

resistance would always be intermediate between that given by the assumption that fluidities are additive and that given by the assumption that viscosities are additive. Such supposition is quite unwarranted, for if in Fig. 1 the surface AB is sheared to the right in respect to the surface CD, the individual lamellæ will tend to become inclined toward the right and very greatly elongated. An early stage of this development is depicted in Fig. 3. Thus although at the beginning the viscosities are strictly additive, as the viscous flow progresses, the conditions change more and more until the *fluidities* are additive. In miscible liquids, the condition that viscosities are additive appears to be very fleeting, disappearing as soon as the liquids are fully mixed. If the substances are immiscible, cohesion may prevent the continued elongation of the lamellæ, in which case the viscosities may be additive. In the case of emulsions, the conditions may be intermediate. This important case will be discussed late.

Experimental Verification of the Theory in Liquid Mixtures.[1]

It now seems extraordinary that investigators have so long held to the belief that viscosities are under all conditions additive, particularly when the experimental evidence gave no support to their belief. To show that this is true, one needs only quote their own conclusions. Dunstan,[2] for example, says: "The law of mixtures is never accurately obeyed and divergences from it seem to be more clearly marked out in the case of viscosity than with other properties, such as refractive index." Thorpe and Rodger[3] go deeper when they say: "The observations described in this paper afford additional evidence of the fact indicated by Wijkander[4] and supported by Linebarger,[5] that the viscosity of a mixture of miscible and chemically indifferent liquids is rarely, if ever, under all conditions, a linear function of the composition. It seldom happens that the liquid in a mixture preserves the particular viscosity it possesses in the unmixed condition. To judge from the instances heretofore studied, the viscosity of the mixture is, as a rule, uniformly lower than the mixture law would indicate, but no simple relation can yet be traced between the viscosity of a mixture and its chemical constituents." They are so confident that there is some uncomprehended principle which causes the viscosity curves to fall below the expected straight line, that they are dubious about some of the viscosity curves of Linebarger which

[1] Cp. third paper.

[2] J. Chem. Soc., *85*, 819 (1904).

[3] *Ibid.*, *71*, 374 (1897).

[4] Lunds, Phys. Sallsk. Jubelschrift, 1878; Wied. Beibl., *8*, 3 (1879).

[5] Am. J. Sci., *11*, 331 (1896).

are very nearly straight lines. They say[1] "The observed viscosities in general are less than those calculated by the mixture rule, except, possibly, in the case of mixtures of carbon disulphide with benzene, toluene, ether and acetic ether, where, possibly, the temperature of observation (25°) was too near the boiling point of the carbon disulphide to make any specific influence, which that liquid might exert at lower temperatures, perceptible." That this assumption was without experimental justification, may be shown by the fact that these same investigators[2] had very carefully measured the viscosities of many liquids, including carbon disulphide, from zero to the boiling-point of each liquid without finding any peculiarity in the viscosity curve at the higher temperatures.

It may be shown that the viscosity of a mixture when the fluidities are additive will generally be smaller than when the viscosities are additive. Equations (1) and (2) represent these respective assumptions and for convenience we will assume that only two components are present. From equation (2) we get that

$$\frac{1}{\varphi'} = \frac{m_1}{\varphi_1} + \frac{m_2}{\varphi_2}$$

or

$$\varphi' = \frac{\varphi_1\varphi_2}{m_2\varphi_1 + m_1\varphi_2}.$$

When $m_1 = 0$ or 1, and $m_2 = 1$ or 0 respectively, φ must be equal to φ'. For all intermediate values of m_1 and m_2 we desire to learn whether φ must be invariably greater than, equal to, or less than φ'. Multiplying equation (1) by unity, we obtain

$$\varphi = \frac{(m_1\varphi_1 + m_2\varphi_2)(m_2\varphi_1 + m_1\varphi_2)}{m_2\varphi_1 + m_1\varphi_2}$$

$$= \frac{(m_1^2 + m_2^2)\varphi_1\varphi_2 + m_1m_2(\varphi_1^2 + \varphi_2^2)}{m_2\varphi_1 + m_1\varphi_2} \gtreqless \frac{\varphi_1\varphi_2}{m_2\varphi_1 + m_1\varphi_2} = \varphi'.$$

Since $m_2 = 1 - m_1$,

$$2m_1(m_1 - 1)\varphi_1\varphi_2 + m_1(1 - m_1)(\varphi_1^2 + \varphi_2^2) \gtreqless 0.$$

Discarding the known roots, $m_1 = 0$ and $m_1 = 1$, we get

$$\varphi_1^2 - 2\varphi_1\varphi_2 + \varphi_2^2 \gtreqless 0,$$

[1] J. Chem. Soc., *71*, 361 (1897).
[2] Phil. Trans., *185A*, 397 (1894).

which is a perfect square and therefore must be positive. Hence when φ_1 is equal to φ_2, φ is equal to φ', but at all other times φ must be greater than φ'. This conclusion may be stated as follows: *The viscosity of a thorough mixture of chemically indifferent liquids must always be less than would be expected on the assumption that viscosities are additive, but this inequality will approach zero as the difference between the viscosities of the components approaches zero. On the other hand, the viscosity of an emulsion* (cp. later) *must be greater than that of a perfect mixture of the same composition.* Thus is explained the irregular drop in fluidity as a mixture is cooled below its critical solution temperature, to be discussed later, as well as the peculiarities of perfect mixtures noted above.

Equation (1) may be expressed in the form

$$\varphi = \varphi_1 + (\varphi_2 - \varphi_1)m_2, \tag{3}$$

where φ_1 and φ_2 are constants and φ and m_2 are variables. The corresponding viscosity equation is

$$\eta = \frac{1}{\varphi_1 + (\varphi_2 - \varphi_1)m_2}. \tag{4}$$

It is very important to note that equation (3) is the equation of a straight line, but that equation (4) is the equation of a hyperbola. If we replace $\varphi_1 + (\varphi_2 - \varphi_1)m_1$ by $(\varphi_2 - \varphi_1)m$ and where

$$m = m_1 + \frac{\varphi_1}{\varphi_2 - \varphi_1}$$

we get

$$m\eta = \frac{1}{\varphi_2 - \varphi_1}$$

which is the equation of an equilateral hyperbola, whose X-axis is at a distance $\varphi_1/(\varphi_2 - \varphi_1)$ to the left of the origin to which equation (4) is referred. From equation (4) we find that the curvature is

$$k = \frac{2(\varphi_2 - \varphi_1)^2[\varphi_1 + (\varphi_2 - \varphi_1)m_1]^3}{\{[\varphi_1 + (\varphi_2 - \varphi_1)m_1]^4 + (\varphi_2 - \varphi_1)^2\}^{3/2}}. \tag{5}$$

By differentiating this curvature in respect to the concentration m_1, and equating to zero, we find the concentration where the curvature is a maximum to be

$$m_1 = \frac{\sqrt{\varphi_2 - \varphi_1} - \varphi_1}{\varphi_2 - \varphi_1}. \tag{6}$$

Substituting this value in equation (5), the amount of curvature, where the curvature is a maximum, is found to be

$$k = \sqrt{\frac{\varphi_2 - \varphi_1}{2}}. \qquad (7)$$

It follows, therefore, as a necessary consequence of the assumption that fluidities are additive that:

1. The curvature obtained by plotting viscosities is greatest when the difference between the fluidities, *i. e.*, $\varphi_2 - \varphi_1$, is large, and becomes zero when $\varphi_2 - \varphi_1 = 0$; cp. equation (7).

2. The curvature obtained by plotting viscosities must continually decrease as the concentration increases unless the square root of $\varphi_2 - \varphi_1$ is greater than φ_1, in which case the point of greatest curvature will be found at some positive concentration [cp. equation (6)]. This latter case will hardly be met with except in the case of undercooled liquids and amorphous solids.

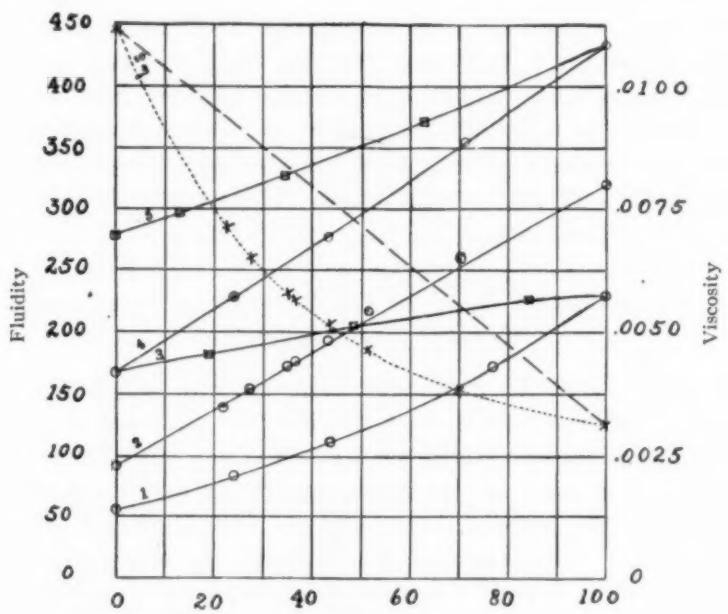


Fig. 4.

1, Fluidity curve of nitrobenzene and ethyl acetate at 25°; 2, fluidity curve of ethyl alcohol and acetone at 25°; 3, fluidity curve of benzene and ethyl acetate at 25°; 4, fluidity curve of benzene and ethyl ether at 25°; 5, fluidity curve of carbon bisulphide and ethyl ether at 25°; 2η, viscosity curve (dotted) of ethyl alcohol and acetone at 25°. Were viscosities additive, this curve would be linear (dashes).

3. Mathematically considered, the curvature is dependent only upon the difference in the fluidities of the components, *i. e.*, $\varphi_2 - \varphi_1$ and not upon φ_1 [cp. equation (7)], but since we can only realize positive values of m_1, it follows that for a given value of $\varphi_2 - \varphi_1$ the curvature at any concentration will be greatest when φ_1 is very small.

The truth of the first conclusion has been observed experimentally by Thorpe and Rodger[1] and White[2] and others.[3] In reference to the second it has been noticed that viscosity curves often pass through a region of very great curvature, the "Erweichungsgebiet." This has been observed[4] particularly of the changes of viscosity of a pure liquid with the temperature, but as will be pointed out later, these curves follow the same laws as do those of mixtures. The third conclusion is verified by the observations of White and others. However all of these peculiarities are characteristic of the viscosity curves only. The fluidity curves are far simpler and must therefore come to be used generally for purposes of physicochemical investigation. In Fig. 4 are given the fluidity curves of mixtures of a number of pairs of substances whose viscosity curves are particularly curved and the viscosity curve of one pair for comparison.[5] It is seen that the fluidity curves are almost linear[6] and hence are almost perfectly defined by their slopes and intercepts. It must be stated here that there are many mixtures which do not give linear fluidity curves, presumably because the components are not chemically indifferent to each other. These cases will be discussed in their appropriate place.

EXPERIMENTAL VERIFICATION OF THE THEORY IN PURE LIQUIDS AT VARYING TEMPERATURES.

It is evident that the reasoning which has been found to apply to mixtures of different liquids must also hold for mixtures of the same liquid at different temperatures; for a liquid at any temperature may be thought of as a mixture of appropriate amounts of portions of the liquid maintained at the extreme temperatures. Hence we are led to the hypothesis that *the temperature-fluidity curves of pure liquids should normally be linear, and the temperature-viscosity curves hyperbolic.* This

[1] J. Chem. Soc. (London), *71*, 361 (1896).

[2] J. Indust. and Engin. Chem., *4*, 270 (1912).

[3] Cp. Bingham, Am. Chem. J., *35*, 195 (1906).

[4] Tammann, Z. physik. Chem., *28*, 18 (1898).

[5] Dunstan, J. Chem. Soc. (London), *85*, 822 (1904); Linebarger, Am. J. Sci., *2*, 331 (1896).

[6] The concentrations were given in weight per cent. We have pointed out that volume percentages should have been used. That better results are obtained by using the later will be shown in connection with the discussion of hydrates, in our next paper.

may not experimentally be realized, for the volume of a liquid does not increase in a linear manner as the liquid is heated, and as will be indicated later, fluidity is apparently closely related to volume; and association and dissociation may play a disturbing effect. In Fig. 5 are given the

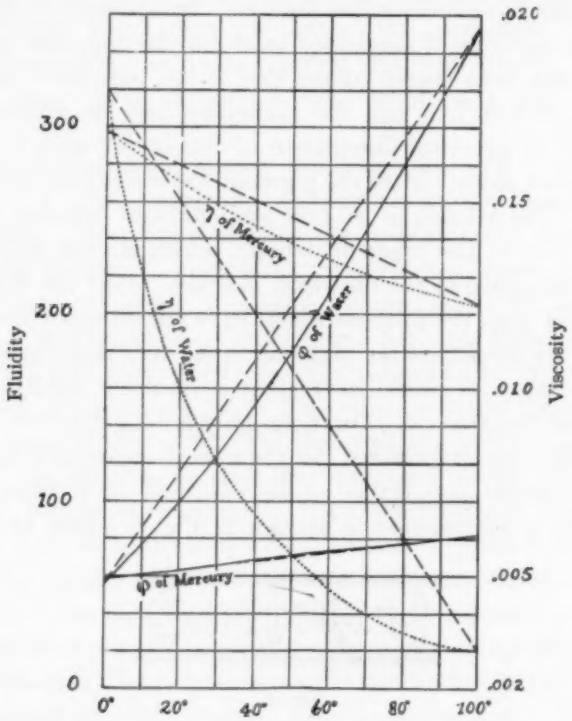


Fig. 5.

Fluidity (continuous) and viscosity (dotted) curves for mercury and water.

temperature-fluidity and temperature-viscosity curves of mercury and water from 1° to 100° C. Both of the former are much more nearly linear than the latter, the true linear curve being represented in each case by a series of dashes. The fluidity curve for mercury is almost absolutely linear, and what curvature there is is in a direction opposite to that of every other known substance, so that it can hardly be regarded as certain that this is not due to experimental error. The most marked deviations from linearity occur in the alcohols, which of course are highly associated, but even they approach linearity at high temperatures.

EXPERIMENTAL VERIFICATION OF THE THEORY IN EMULSIONS, SUSPENSIONS, AND COLLOIDAL SOLUTIONS.[1]

Emulsions, suspensions, and colloidal solutions may profitably be considered together. Emulsions differ from the others in that both of the components have measurable fluidity. In suspensions and colloidal solutions the particles of suspended solid may be supposed to have zero fluidity. It has been shown above that if the individual drops of the emulsion are of sufficient size the viscosities may be additive and not the fluidities. A practical illustration of this is met with in stirring up a pail of "mixed paint," after the pigment has settled out on the bottom of the pail. The stirring is at first very difficult because the viscous resistance is that of the pigment deposit, which is very great, added to the viscous resistance of the oil which is small. Had the fluidities been additive in this case, the mixture would have stirred easily from the first, since adding the fluidity of the pigment to the very much greater fluidity of the supernatant oil would have given a resultant fluidity nearly the same as that of the oil alone. On thoroughly mixing the paint, the fluidity is raised (or the viscosity lowered) for the viscosities are no longer additive.

The hypothetical appearance of an emulsion as it flows through a capillary tube is represented in section in Fig. 6. Due to the friction

Fig. 6.

against the walls, the front end of the globules tend to become more convex and the rear end less convex than would be the case were the globules stationary. It is specially to be noted that when the globules are small in diameter and yet large enough to fill the whole cross-section of the tube, the motion of the liquid is not entirely linear, it being transverse as well as horizontal, as indicated by the arrows in the figure. Were the motion perfectly linear, we have already proved that the viscosities would be strictly additive, and that the resultant viscosity would be greater than would have been the case if the components had formed a perfect solution. But the effect of the transverse motion is to still further increase the viscosity. If, however, the drops are very large in comparison with the diameter of the tube, the effect of this transverse motion may be rendered negligible. It is self-evident therefore that if the globules of an emulsified liquid are large enough to fill the cross-

[1] Cp. eleventh and twelveth papers.

section of the transpiration tube, the viscosity will be at least as great as the sum of their viscosities multiplied by their respective volume concentrations [cp. equation (2)]. It may be greater due to the transverse motion. It may be remarked in passing that miscible liquids, when the mixing is excessively imperfect may show a resultant coefficient of viscosity which approaches the sum of the component viscosities; but it can never exceed that value because there can be then no transverse motion so long as the pressure is low enough to permit ordinary linear flow.

The above conclusion is very unambiguous, and mixtures of liquids which have a critical-solution temperature seemed best suited for the experimental confirmation because the values of the fluidity when the components formed an emulsion or a perfect solution could both then be determined at nearly the same temperature. On looking up the literature, it was found that Ostwald, and Stebutt,[1] Friedländer,[2] Scarpa,[3] Rothmund,[4] and Bose[5] had all found abnormally large viscosities in lowering the temperature below the critical-solution temperature. But they had none of them perceived that this was to be expected from the nature of viscous flow, so that they proposed several different theories to account for the phenomenon. For a discussion of these theories, we need only refer to a paper by White and the present writer.[6] In Fig. 7 we give the fluidity curves for phenol and water mixtures taken from their paper. The data were partly determined experimentally and partly taken from the work of Scarpa and recalculated to a fluidity basis. It is seen that there is always a tendency for the fluidity to fall off abnormally as the critical-solution temperature is approached. The results are irregular as are those of some of the earlier workers, and this is to be expected since the viscosity is to some extent at least dependent upon the size of the drops. It was noted by Bingham and White that the abnormality in the fluidity curves began before the critical-solution temperature was actually reached, *i. e.*, in the opalescent condition of the liquid. It becomes necessary for us therefore to point out that the decrease in fluidity does not depend upon the formation of actual drops but merely upon a sufficient lack of homogeneity in the liquid. The opalescence itself is, we believe, an indication of the lack of homogeneity,

[1] Lehrbuch der allgem. Chem., 2 Aufl., 2^2, 684, etc.

[2] Zeitschr. f. physik. Chem., *38*, 385 (1901).

[3] Cimento [5], *5*, 117 (1903), and J. Chem. phys., *2*, 447 (1904).

[4] Zeitschr. f. physik. Chem., *63*, 54 (1908).

[5] Physik. Zeitschr. *8*, 313, 347 (1907); *9*, 769, 707 (1908); *10*, 32, 230, 406 (1909); Zeitschr. f. Elektrochem., *8*, 449 (1907).

[6] J. Amer. Chem. Soc., *33*, 1257 (1911).

and it is not very difficult to explain. At the critical-solution temperature the affinity of the molecules of phenol for other molecules of phenol added to that of molecules of water for other molecules of water becomes

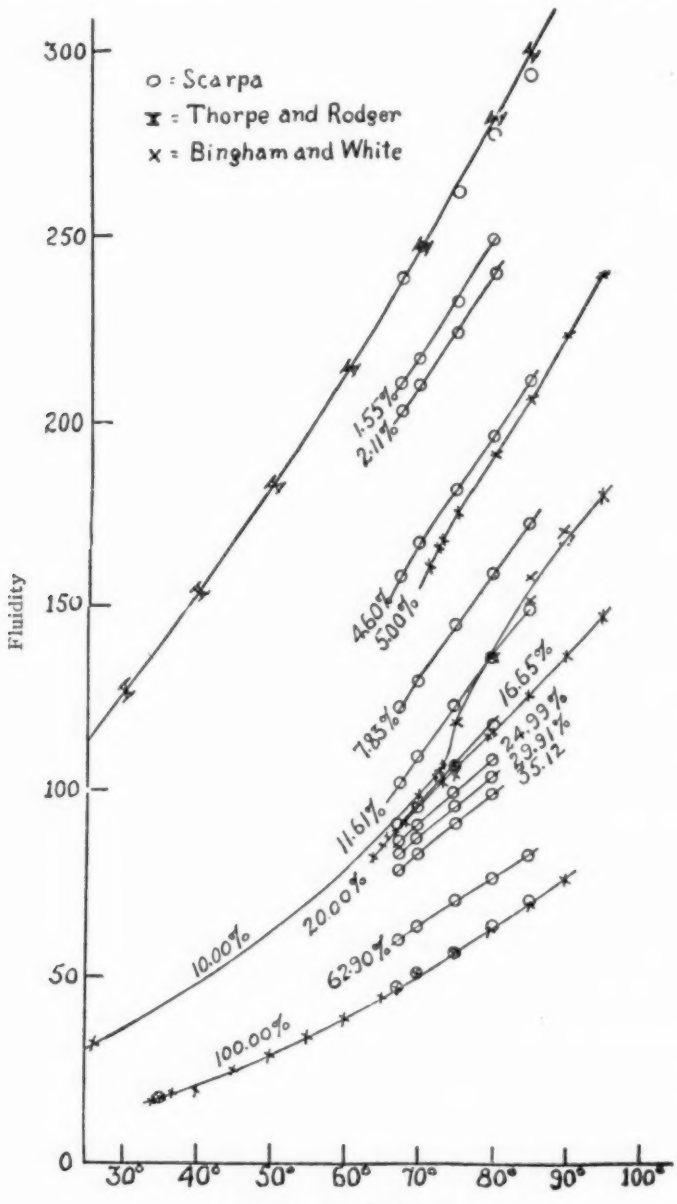


Fig. 7.

The fluidities of phenol and water mixtures.

equal to the affinity between the phenol and water. At the critical temperature, any slight changes in temperature have a maximum effect. But even at temperatures considerably above the critical-solution temperature, some of the phenol molecules and some of the water molecules must have velocities below the critical value. The result will be that even above the critical-solution temperature the molecules of phenol will group themselves together, and the molecules of water will likewise group themselves together, and if they reach sufficient size before being redissolved, the action of gravity will be appreciable in causing a separation of the two components.

It is evident that the same lack of homogeneity will explain the increase in viscosity in crystalline or " anisotropic " liquids. It is only necessary that the " swarms " be of sufficient size to make the viscosities additive.

We have seen that in perfect solutions the fluidities are normally additive. In emulsions the viscosities may be additive, the exact value depending upon the conditions of flow. When the suspended particles are very small as in most suspensions, it is not self-evident whether one or the other assumption is correct, or whether the value should be intermediate between the two. To clear up this point let us consider some simple cases.

If we imagine the solid particles of a suspension to be united into sheets parallel to the direction of flow, like the shaded areas in Fig. 1 when the surface *CA* is sheared upward in respect to *DB*, then it is evident that, provided that there is no slipping at the boundary between the liquid and solid, the flow will be the sum of the flows of the unshaded areas, *i. e.*, the fluidities will be strictly additive or

$$\varphi = \varphi_1 m_1 + \varphi_2 m_2 = \varphi_2 m_2, \tag{8}$$

since φ_1 is practically zero.

But if these solid sheets were broken into small fragments, one of which is shown as the cross-section of a cube at *F* in Fig. 8, the de-

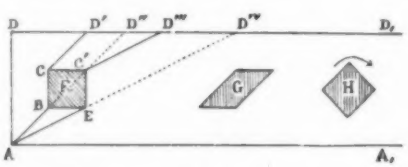


Fig. 8.

formation of the liquid would tend to change the cube into a parallelopiped shown at *G*, but as the solid is rigid, this cannot take place; so that the

shearing force can only rotate the cube around its center as shown at *H*. But the failure of the solid to change its shape with the flow of the liquid. will necessitate transverse motions in the liquid by way of readjustment, hence the viscosity of a suspension will always be greater than it would

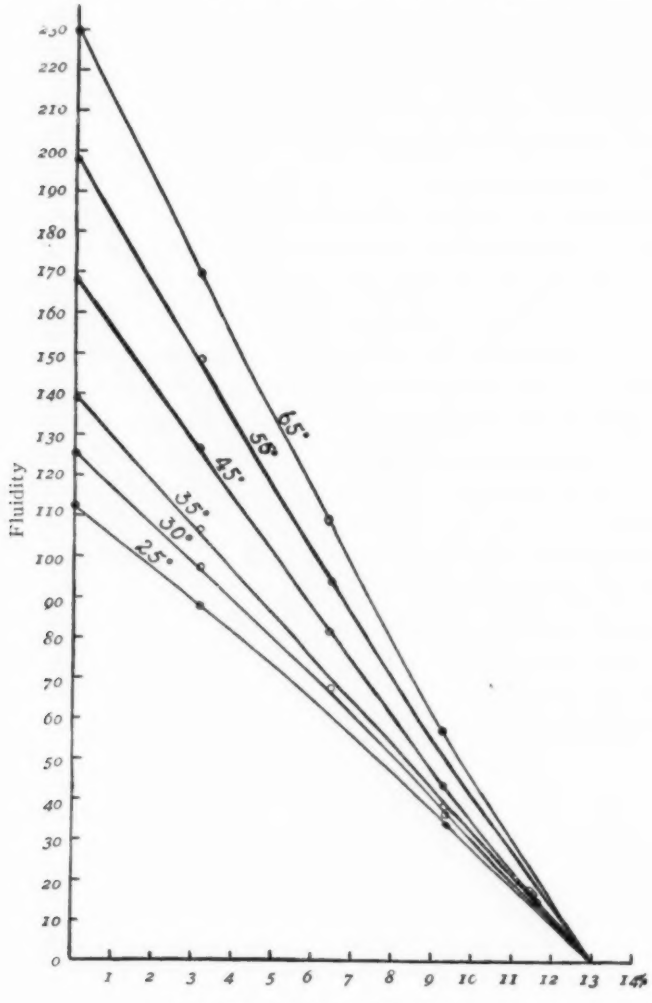


Fig. 9.

The fluidity of suspensions of infusorial earth in water at different temperatures and concentrations.

be were the fluidities strictly additive. This conclusion would not be true however if there were slipping at the boundary. Since if the viscosities were additive, the viscosity of a suspension would be infinite

which is certainly not always found to be the case, we reach the conclusion that *the fluidity of a suspension must always be intermediate between that calculated on either of the two fundamental assumptions*, equations (1) and (2).

A different way of looking at this question will be given later in discussing the behavior of a liquid at the boundary, which leads to the same conclusion. We will first give the fluidity curves of some suspensions of infusorial earth, china clay, and "Aquadag" in water and alcohol, as determined by Durham and the writer,[1] cp. Figs. 9–13. It is observed

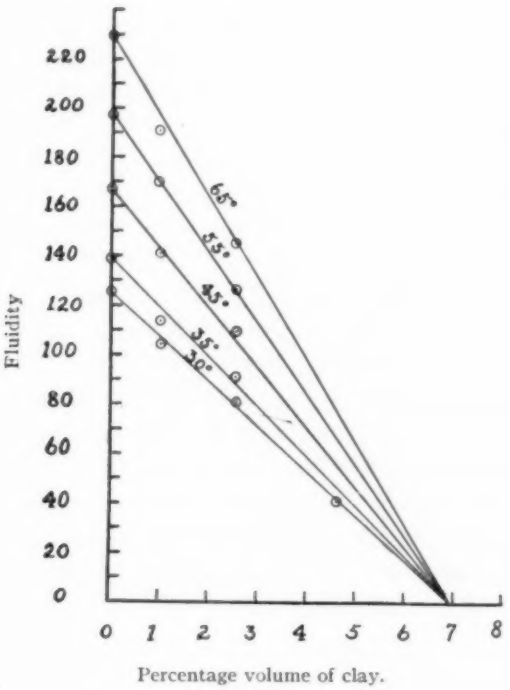


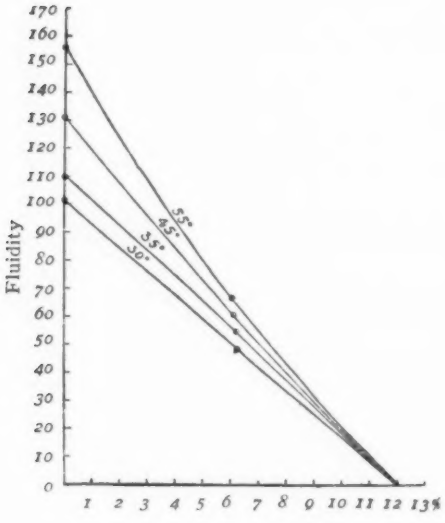
Fig. 10.

The fluidities of suspensions of English china clay in water at different temperatures and concentrations.

that *as the concentration of the solid present in a suspension rises, the fluidity of the suspension rapidly decreases and in a linear manner, so that at no very high concentration of solid, the fluidity would be zero, and even more extraordinary is the fact that this zero of fluidity is quite independent of the temperature.* Whether very greatly different pressures or other methods of viscosity measurement would affect these curves has not been

[1] Amer. Chem. J., *46*, 278 (1911).

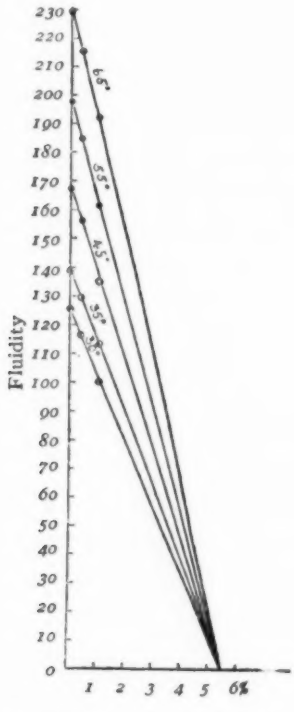
shown with certainty. It is of great importance that this point should be established.



Percentage volume of earth.

Fig. 11.

The fluidity of suspensions of infusorial earth in ethyl alcohol at different temperatures and concentrations.



Percentage volume of graphite.

Fig. 12.

The fluidity of suspensions of graphite-Aquadag in water at different temperatures and concentrations.

This zero of fluidity in suspensions has apparently not heretofore been noticed. And it speaks very much in favor of the conceptions which we have been advancing that, with the old idea that only viscosities are additive, the remarkably simple relations which suspensions are here seen to exhibit, could scarcely have been discovered at all since the viscosity curves would only meet at infinity. But that these suspensions can have a zero of fluidity seems at first quite incomprehensible, because a suspension with zero fluidity is still sufficiently mobile so that if a bottle containing a considerable quantity of the mixture is inverted the material will run out. But *this* flow cannot be viscous flow, it must be plastic flow. This "zero of fluidity," we therefore believe to be the point of demarcation between viscous and plastic flow, the distinction being that in the former any shearing force, however small, will produce continuous

deformation, while in the latter a definite minimum of shearing force must be applied before continuous deformation will take place. If this explanation proves to be correct, it seems clear that the zero of fluidity ought to be dependent upon the pressure; for if sufficient pressure is at our command it seems probable that even the solid itself will show some degree of viscous flow. The fluidity formula of suspensions [cp. equation (8)] is

$$\varphi = \left(1 - \frac{m_1}{c_1}\right)\varphi_2, \tag{9}$$

where m_1 and m_2 are the volume concentrations of the suspended substance and the medium respectively, and c_1 and c_2 are their particular values where the fluidity becomes zero. Comparing this equation with equation (8) $\varphi = (1 - m_1)\varphi_2$ we see that, since c_1 is always less than unity, the effect of the transverse motions brought about by the presence of the solid particles is merely to increase the apparent concentration of the solid. It appears as it would if each particle of solid lowered the fluidity of the liquid immediately surrounding it to a negligible quantity, so that when the whole space is bridged over by these effects, the whole mixture may be said to possess " zero fluidity."

It will help in understanding this phenomenon to enquire into the behavior of a liquid near its boundary of solid. Helmholtz and Piotrowski[1] have given the results of experiments which indicate that slipping is considerable. The great prestige of Helmholtz made experiments which failed to show slipping carry less weight than would have been the case if the opposite conclusion had had a lesser champion. Hence a large number of somewhat elaborate experiments have been carried out by various experimenters[2] in order to decide the point. Ladenburg[3] repeated with great care the actual experiments of Helmholtz and Piotrowski but without finding any evidence in favor of slipping. It therefore seems to be conclusively proved that perceptible slipping does not occur at the boundary between liquid and solid at least provided that they wet the solids over which they move.

And theoretically it seems most improbable that slipping should occur in any case at the boundary between a liquid and a solid, not only because of the inequalities in the ultimate surface of all solids, but because, with the high adhesion which is known to exist between all solids and liquids whether the solids are wet or not, it seems improbable that the flow in

[1] Wien. Ber., *40* (*2a*), 607 (1868).

[2] Whetham, Proc. Roy. Soc. (London), *48*, 225 (1890); Hadamard, Compt. rend., *136*, 299, 545 (1903).

[3] Ann. Phys. [4], *27*, 157 (1908).

the very thin layer in contact with the solid should be appreciable in comparison with the flow in the practically infinite number of layers of liquid which are moving over each other.

It is usually assumed that the layer of liquid in immediate contact with the solid is absolutely stationary, but it is much more logical to admit that there may be flow, and that it is subject to the well-known laws of viscous flow. The fundamental law of all linear viscous flow is that the rate of flow at any point in a homogeneous liquid in respect to a solid surface regarded as stationary is directly proportional to its distance from that surface. It is upon this fundamental law that all of our conceptions of viscosity ultimately depend. It tells us that velocities in a liquid increase in an arithmetical progression, so that the rate of flow near the boundary is of necessity very small, and in the flow of a liquid through a capillary tube it is well-known that a decrease in the diameter of only one half, causes a diminution in the volume of flow to one sixteenth of its former value. This result is so commonplace that its full effects have been generally overlooked. As an example it may be cited that lampwicking is woven so tightly that kerosene is not easily drawn up by the capillary forces which at these microscopic distances are very powerful. So the fabric is treated with steam which expands the pores. But with a less fluid substance like hot pitch or various gums the velocity of flow would still be negligible, nor can it be made to rise by still further enlarging the pores, because then the capillary attraction is greatly weakened. Hence when goods are to be saturated with a gum like rubber, a solvent is absolutely necessary in order to raise the fluidity to a workable quantity. Each of the solid particles of a suspension is, according to this explanation, surrounded by a layer of liquid which is practically stationary in reference to the particle—not absolutely so—and this gives rise to the zero of fluidity in a mixture with comparatively small percentage of solid present. As the number of particles in the suspension increased, this stationary space is also increased in proportion, *i. e.*, these stationary spaces do not overlap, because as two particles approach each other, they will not quickly penetrate each other's stationary spheres, when there is no strong specific attraction between the particles. In other words, the fluidity is lowered in a linear manner.

Thus may be explained why a suspension will often settle out quickly so as to give two distinct layers, but only after a very considerable time has elapsed will the layers of solid particles sink to their final volume, the particles approaching each other finally with infinite slowness. The same principle is involved in the clogging of a filter by means of fine

precipitates. The pores in the fine precipitates are smaller than in coarse-grained precipitates, and the smaller diameter of the pores very much more than makes up for their greater number, their area being the same in both cases as can easily be proved. Furthermore every chemist knows that if the suspended precipitate is poured on to the filter before wetting the paper, the trouble from clogging is increased. This is doubtless because the suspended particles are forcibly drawn into the pores of the paper by the powerful capillary forces, and then to make matters worse, the pores are partly closed up by the expansion of the cellulose on becoming wet.

Perhaps the most interesting question in regard to the fluidity of suspensions arises in regard to the effect of the size of the particles, for we have seen that the zero of fluidity is considerably different for suspensions of different materials. Since the substances used were insoluble in the liquids employed, one naturally looks for an explanation in the differences in the sizes of the particles. The explanation for the zero of fluidity which we have offered, would seem to demand that the fluidity decreases as the size of the particles is reduced, since thereby their surface is enlarged. This decrease in fluidity will however not continue indefinitely, because when the particles approach molecular dimensions, their vibrational velocity will become appreciable and in liquids the increase in vibrational velocity is associated with increased fluidity. Ordinary experience is in harmony with this inference for it tells us that the fluidity of a liquid is not greatly decreased by the presence in it of an amount of very coarse material, of nearly the same specific gravity as the liquid, which would very greatly decrease the fluidity if the particles were in a very fine state of subdivision. Aquadag must be very finely divided since it will stay suspended indefinitely, yet it depresses the fluidity more than any other suspension which has been studied by us. It is well known that very small amounts of colloids exert a very great effect in depressing the fluidity. The experiments of Woudstra[1] on the viscosity of colloidal silver solutions indicate that a zero of fluidity would be reached at a concentration of considerably less than one volume per cent. of colloidal silver.

Very fortunately for the views here presented, Woudstra has observed that colloidal silver solutions increase in fluidity on standing, and also on the addition of electrolytes. He recognizes that in either case coagulation is favored and he is thereby led to see that viscosity is here dependent upon the area of the boundary surface. He furthermore recognized that colloidal silver solutions are very different from colloidal

[1] Zeitschr. f. physik. Chem., *63*, 619 (1908).

solutions of gelatine and silicic acid, for with the latter the fluidity decreases with coagulation. Happily even this latter class of solutions does not speak against the views here presented as their behavior is readily explained. Many chemists believe that in a gelatine solution a "liquid precipitate" separates out as the coagulation proceeds, which forms a network through the liquid. If this is true, the fundamental law of viscous flow cannot be realized in that the rate of flow will not increase in direct proportion to the distance from the boundary of the mixture. The network of precipitate will extend the region of no flow out into the liquid, and the flow which takes place through the pores of the precipitate will be very slight. It is common experience that a gelatine solution after it has thoroughly set may be rendered considerably more fluid by simply cutting it in various directions or by grinding it in a mortar or by shaking. It is self-evident from what has been already said, that cutting the network of precipitate will increase the fluidity. Perhaps however there is no sharp line of demarcation between the fine compact precipitate of silver and the voluminous precipitate of gelatine.

Our discussion of the effect of the size of the particles upon the fluidity of the suspension would be incomplete, if we left the impression that the zero of fluidity is independent of the nature of the suspending liquid. In the curves already given, it is to be noticed that the zero of fluidity for infusorial earth is not quite the same in alcohol that it is in water. But far more extraordinary are the effects produced by the addition of small amounts of electrolytes to the suspensions in water, and contrary to the usual impression, the electrolytes are very diversified in their effects. For we have observed that the addition of small amounts of any substance known to give hydrogen ions invariably produces a drop in the fluidity of the suspensions with which we have worked. On the other hand, small amounts of substances known to give hydroxyl ions, cause an increase in fluidity, and salts which give a neutral reaction give a mixed effect. Moreover the addition of an acid to these suspensions produces marked coagulation. The coherence of the particles together in gelatinous masses causes the formation of networks throughout the liquid which probably lower the fluidity after the manner already explained for gelatine solutions.

It is not at all essential to our theory of the relation of fluidity and viscosity to each other and to other properties that we understand the cause of the increased coherence which is manifest in the presence of hydrogen ions, but the question is an interesting one and undoubtedly its complete answer would suggest further opportunities for testing our theory. Hence we shall state some of the reflections which our work up to the present time suggests.

The action of soap in cleansing has usually been attributed to the action of the hydroxyl ions in causing the saponification of the grease. But this hypothesis has always seemed inadequate and it fails entirely to explain the equally high efficiency of soap in removing mineral oils and greases. To say that the soap promotes the emulsifying of the oil or grease is a satisfactory restating of the phenomena but it is scarcely adequate as an explanation. The procedure of a workman in removing mineral grease from his hands is usually as follows. He first pours kerosene, gasoline, or other fluid oil upon his hands and mixes the grease and oil together by rubbing. Thus the fluidity of the grease is raised. With a little water and soap he then produces a thick lather on his hands; and incidentally, although vital to the success of the operation, the rubbing and consequent viscous flow involved in raising the lather loosens the grease, which has already been rendered fluid by means of the oil, and since the particles do not readily cohere in the alkaline solution they do not again become attached to the skin. To complete the operation it is therefore only necessary to wash off the lather. That *any* solution containing hydroxyl ions will be just as efficient as the soap solution with which it is isotonic cannot of course be maintained. But it is certainly true that many substances with an alkaline reaction, like borax, soda, and ammonia have marked cleansing power and that the addition of acid substances to soap solutions entirely destroys their cleansing power, the grease immediately re-attaching itself to the objects being cleaned. I am informed that a liquid soap is being used which consists essentially of an emulsion of a mineral oil in a dilute solution of caustic soda.

It appears that *emulsions and suspensions are generally more stable in alkaline than in acid solutions.* Many illustrations of this might be given, but a few must suffice. India ink is immediately coagulated by a trace of acid and the same is true of the suspension of carbon known as "Aquadag." Mercury can be easily emulsified, and it is customary to get it back again by boiling with hydrochloric acid. We have found that the Richmond deposits of infusorial earth may be elutriated and a portion of material obtained which will remain in suspension for many days. But this suspension is rapidly coagulated by an acid solution and the elutriation cannot be successfully carried out in acidulated water. In contrast with this, we have noted that English china clay, when stirred up in water, settles out quickly, leaving a clear supernatant liquid, while the material settling out is noticeably flocculent. On adding a little alkali, the coarser materials settle our first, and the liquid at the top becomes only very gradually clear. Moreover the flocculent

character of the deposit is destroyed, which is evidence of the loss of coherence between the particles. With the use of a ball mill clay may be brought into permanent suspension without the use of alkali. But the liquid becomes alkaline automatically on account of the hydrolysis of the feldspar as soon as the layer of insoluble aluminum silicate is removed by the mill from the surface of the particles. Every reader knows that when cream is sour, be it ever so slightly so, the coagulation is very evident as soon as the cream is poured into the morning cup of coffee. The writer is not at liberty to cite some important illustrations of this principle arising in the industrial arts. It is probably true that only the technical chemist, who habitually resorts to dilute alkali and a ball mill for getting substances into emulsion or suspension, can truly appreciate the generality of this principle.

Perhaps the illustrations given, together with others which will readily occur to the reader, will be sufficient to show that the coagulation in the presence of hydrogen ions is a somewhat general phenomenon, and that it is due to the increased coherence between the particles; but they are not sufficient to give a clear insight into the cause of this increased cohesion. Perhaps the following illustration will help to do this. When a solution of rosin in alcohol is thrown into water, a suspension is formed which is quite permanent since I have found that it would remain for many weeks without settling out. However on adding a drop of a dilute solution of a strong acid, the whole suspension is beautifully coagulated, clouds of flocculent precipitate floating about in the clear liquid. But on adding an equivalent amount of alkali the suspension is again formed nearly as at first, and on the addition of more the insoluble acid begins to dissolve. This change is so sharp and so readily detected that it is entirely possible to use such a suspension as an indicator in acidimetry and alkalimetry. It appears from this instance that the thinnest sort of a layer of the soluble salt, such as may occur in what is usually termed a neutral solution is sufficient to prevent the coherence of the particles. But as soon as a drop of acid is added this thin layer of salt is destroyed leaving a fresh surface of the insoluble acid. But while the particles of acid are not soluble in water they are soluble in each other,[1] hence cohesion according to the well-known phenomena of surface tension pulls the particles together and holds them. But as alkali is added the solid particles are again covered by a layer of solution which is soluble in water and perhaps at the same time in the free acid hence the particles are no longer held together. So the phenomena are to be explained by the difference in solubility between the alkali salt and the free acid. It

[1] Cp. article by the writer on "Solubility," Amer. Chem. J., *38*, 91 (1907).

is true of abietic acid as of the greater number of organic acids that it is much more insoluble in water than are its alkali salts, hence the explanation of this case is capable of rather extended application. Of course it is not impossible according to this explanation that there may be cases where the addition of alkali will cause coagulation and a decrease in fluidity, but we do not happen to have discovered such. Perhaps we should consider a solution of ferric chloride to be an illustration of such a case, since here we have a colloidal solution of ferric hydroxide which is readily coagulated by the addition of alkali.

Having discussed the effect of the size of the particles in a suspension upon the fluidity of the mixture and also the extraordinary effect of small differences in the chemical character of the suspending medium, it becomes important to ask how the fluidity of the mixture is dependent upon the fluidity of the medium. To decide this, we may refer to Fig. 8, where the lower surface AA_1 is supposed to be stationary while the upper surface DD_1 is sheared to the right under a given force, such that the point D would be sheared to D'' in a unit time. The distance of any point in the line AD'' from the line AD represents the velocity in its respective layer, the motion being assumed to be low enough so that it is not turbulent. But if the layers from B to C are replaced by a continuous solid and if there is no slipping at the boundary, the velocity of flow will be constant from B to C and the curve representing the velocities will become $ABCD'$. There is a loss in velocity (or fluidity) of $CC' = D'D''$, which is proportional to the amount of liquid replaced by solid, so that

$$\frac{D'D''}{DD''} = \frac{BC}{AD} = m_1.$$

With another liquid, but the other conditions the same, the velocity at the surface DD_1 might be DD^{IV} but the ratio BC/AD would be the same, so we reach the conclusion that *the reduction in the fluidity of a liquid by the addition of a given quantity of finely divided solid is directly proportional to the fluidity of the liquid* provided, of course, that solution or agglutination do not occur. This may also be deduced directly from equation (8). As we have seen, suspensions are somewhat more complicated than the above case on account of transverse motions, but the law applies to these transverse motions as well.

We have then a test of our theory which was for a long time overlooked by us. The law states that the reduction in the fluidity on adding infusorial earth to either alcohol or water must be in direct proportion to

their respective fluidities. Now at 55° C. the fluidities of alcohol and water are considerably different, being 156.5 and 197.8 respectively. But the theory demands that *the volume concentration of the mixture having zero fluidity should be the same in all media.* As a matter of fact the zero fluidity occurs at 12 volume per cent. in alcohol and 13 volume per cent. in water, which is satisfactorily close when we take into consideration the effects of small amounts of impurities and the different dissociating power of alcohol and water.

But the same medium has very different fluidities at different temperatures, and the theory demands that the reduction in fluidity will be in direct proportion to the fluidity which the medium possesses and therefore that *the volume concentration of the mixture corresponding to zero fluidity should be absolutely independent of the temperature.* This is observed experimentally to be the fact within the limits of experimental error.

CONCLUSIONS.

We have thus far given a summary of the theoretical and experimental evidence for the following conclusions:

1. The assumption, which seems to have been universally held, *viz.*, that viscosities are always additive, is quite erroneous.

2. Instead of viscosities being additive, it is much nearer the truth to state that fluidities are additive, which implies that viscosities are hyperbolic. But without careful definition this generalization is both insufficient and misleading. Therefore we add:

3. The fluidity of a thorough mixture of miscible and chemically indifferent liquids is a linear function of the volume-concentration, equation (1). Thus is explained the fact which has been repeatedly observed and not explained, that the viscosities of such mixtures are generally less than would be expected upon the current assumption.

4. The fluidity-temperature curve of pure liquids, or of thorough mixtures of miscible and chemically indifferent liquids, are normally linear. This conclusion is perfectly true for only a very few liquids, *e. g.*, mercury, but it is approximately true for all unassociated liquids and for associated liquids at high temperatures at least.

5. In special cases, the viscosities may be additive, as in either miscible or immiscible though chemically indifferent liquids, when the mixture by solution or emulsion respectively is very incomplete.

6. If the globules of liquid in an emulsion are small and well distributed throughout the mixture, the fluidity may apparently be either less (due to transverse motions) or greater (due to the approach to conditions

where the fluidities are additive) than would be the case if the viscosities were strictly additive (cp. 5), but the fluidity will always be less than it would be, if the fluidities were additive. This may be due to the cause given under 9.

7. In suspensions of finely-divided but insoluble solids, the fluidity will always be less than it would be if the fluidities of the components were strictly additive. As a matter of experimentally determined fact, the fluidity decreases in a linear manner so that at no very great volume concentration the mixture appears to have zero fluidity. The concentration of the mixture having zero fluidity appears to be independent of the nature of the solid or of the suspending medium provided that they do not result in the solution or agglutination of the solid particles; it is independent of the temperature of the medium; and it apparently depends solely upon the size and number of the suspended particles in a unit volume.

8. This mixture of zero fluidity apparently demarcates for the first time viscous from plastic flow.

9. Each particle of suspended solid behaves as if it not only possessed zero fluidity itself, but as if it is able to communicate this to a spherical shell of the liquid surrounding it, the thickness of this shell being apparently thicker at high temperatures than at low ones. This curious behavior is exactly what would be expected if there were no slipping at the boundary between solid and liquid, and is a consequence of the fundamental law of viscous deformation, that the velocity of flow at any point varies directly as the distance of the point from the boundary of the liquid. The greater the fluidity of the medium or what amounts, in general, to the same thing, the higher the temperature, the more pronounced is this effect.

10. As a further consequence of the above conclusion, it follows that the greater the area of solid surface exposed, either by increasing the concentration or by diminishing the size of the particles and thereby increasing their number, the greater will be the diminution in the fluidity of the medium. Suspensions of very fine matter and colloidal solutions afford satisfactory confirmation of this view.

11. In gelatinous precipitates and certain colloidal solutions like that of gelatine where the particles cohere together, forming a network, the solid boundary is projected out into the liquid, hence the fluidity of the mixture is exceedingly low.

12. The addition of electrolytes to suspensions and colloidal solutions produces marked effect upon the fluidity. When coagulation occurs, the fluidity will be increased on account of the reduction in the area of

boundary surface, but on the other hand the fluidity will be decreased if the coagulating particles cohere together in such a way as to form networks through the liquid; so the resultant effect may be either to increase or to decrease the fluidity.

13. In suspensions, acids generally cause gelatinous coagulation and a decrease of fluidity. Alkalies have the opposite effect and neutral salts have a mixed effect. This effect is due to the increased cohesion between the solid particles in a slightly acid solution; and this in turn is interpreted as due to the greater insolubility of most organic acids than of their alkali salts. The greater the insolubility of the acid in water, the greater will be its solubility in itself, *i. e.*, its cohesion.

This cohesion as well as some of the others is capable of considerable practical application. Thus the theory of the detergent action of soap is for the first time made thoroughly rational.

14. Colloidal silver solutions[1] show a decrease in fluidity on standing or on the addition of an electrolyte. This is interpreted as due to the reduction of surface by coagulation, the precipitated colloid being in this case probably crystalline and not gelatinous.

15. Further conclusions in regard to the character of viscosity curves in general have been given on page 413 and need not be repeated here.

The majority of workers still cling to the old hypothesis that viscosities are always additive, hence we have tried to make the argument as strong as possible to show that the hypothesis is untenable. It may be possible for the reader to give his assent to the general argument which is here advanced, and yet it has been true in the past that in applying it in isolated cases, the worker becomes easily confused because of the complexity of many of the actual cases. For example, almost all aqueous solutions give fluidity curves which are not linear. We hold however that this cannot be used to argue for either the truth or the falsity of the theory here proposed until we come to an understanding of the question of hydrates, which undoubtedly may, and probably does, here cause complication. In scrutinizing the evidence for or against any theory, it seems unthinkable that any evidence should be set aside except for a valid reason, and then it is to be set aside only temporarily, until the cause of the complication can be satisfactorily studied. At such a time, these excluded cases may be used as very important evidence for or against the theory.

In the second part of this paper, we shall regard the evidence already given for the theory of viscous flow here presented as conclusive; we shall therefore attempt to throw light upon the nature of the relations between

[1] Woudstra, *loc. cit.*

fluidity and other properties, using in particular those cases which have heretofore been excluded. If the reader feels that the evidence at any point is inconclusive, he is asked to note that since the evidence is cumulative he may well withhold judgment because the remaining part of this paper carries weighty evidence for or against the theory.

RICHMOND COLLEGE,
RICHMOND, VIRGINIA,
September 17, 1912.

DETERMINATION OF CAPACITIES BY MEANS OF CONJUGATE FUNCTIONS.

BY J. W. WOODROW.

THIS paper is intended to make use of the properties of conjugate functions for the solution of a few problems. The cases to be considered are those in which the surfaces of electrical conductors are generated by the motion of straight lines all parallel to a straight line which will be taken as the axis of z. These conductors are to be considered sufficiently long in this direction so that when charged, the z-component of the electric force may be neglected. Then if that portion of the field between two planes parallel to the xy-plane and unit distance apart be considered, the potential and distribution of electricity become functions of x and y only. If a relation can be found between w and z where

$$w = u + iv \quad \text{and} \quad z = x + iy,$$

which will transform straight lines in the w-plane into curves in the z-plane which coincide with the lines of intersection between the equipotential surfaces and a plane perpendicular to them, expressions for the capacity of the system and for the surface distribution can be easily found.[1]

In the following transformations, the equipotential surfaces will be given by u equal to a constant and the lines of force by v equal to a constant. The charge e will indicate the charge on unit length of the conductor and the capacity C will represent the capacity per unit length.

Consider first the simple case of the surface distribution on a very long wire parallel to an infinite plane conductor at zero potential. It is well known that the transformation

$$w = \log \frac{z + a}{z - a} \tag{1}$$

will give the equipotential lines and lines of force for the above condition. The capacity as given by this transformation will be found in Webster's Electricity and Magnetism, where it is also shown that the surface dis-

[1] Electricity and Magnetism, J. H. Jeans, p. 256; Electricity and Magnetism, A. G. Webster, p. 307.

tribution for any transformation by means of the complex variable is given by

$$\sigma = \frac{1}{4\pi}\left|\frac{dw}{dz}\right|. \tag{2}$$

By differentiating (1)

$$\frac{dw}{dz} = \frac{2a[(a^2 - x^2 + y^2) + 2ixy]}{(a^2 - x^2 + y^2)^2 + 4x^2y^2},$$

and hence

$$\left|\frac{dw}{dz}\right| = \frac{\sqrt{4a^2[(a^2 - x^2 + y^2)^2 + 4x^2y^2]}}{(a^2 - x^2 + y^2)^2 + 4x^2y^2}. \tag{3}$$

Whence

$$\sigma = \frac{a}{2\pi\sqrt{(a^2 - x^2 + y^2)^2 + 4x^2y^2}}. \tag{4}$$

Now a more convenient form is obtained by changing to polar coördinates with the origin at the center of the wire. If r is the radius of the wire and d is the distance of its center from the infinite plane which is represented by the y-axis, the following relation is easily deduced:

$$a^2 = d^2 - r^2.$$

Substituting this value of a in (4) and making the transformation to polar coördinates, the expression for the surface density becomes

$$\sigma = \frac{\sqrt{d^2 - r^2}}{4\pi r} \cdot \frac{1}{r\cos\varphi + d}. \tag{5}$$

This gives a minimum value of σ for $\varphi = 0$ and a maximum value for $\varphi = \pi$. The surface distribution on the infinite plane is obtained by putting $x = 0$ in (4) which then becomes

$$\sigma = \frac{a}{2\pi(a^2 + y^2)}. \tag{6}$$

WIRE PARALLEL TO TWO PLANES INTERSECTING AT RIGHT ANGLES.

The equipotential lines and lines of force about a long wire parallel to the intersection of two perpendicular infinite planes are given by the transformation

$$w = \log\frac{z^2 - \beta^2}{z^2 - \alpha^2}, \tag{7}$$

where

$$\alpha = a + bi \quad \text{and} \quad \beta = a - bi.$$

The curves in the z-plane for u equal to a constant are shown in Fig. 1. Near the point (a, b) the curves approximate very closely to circles so that the equipotential line C in Fig. 1 can be replaced by a wire of circular

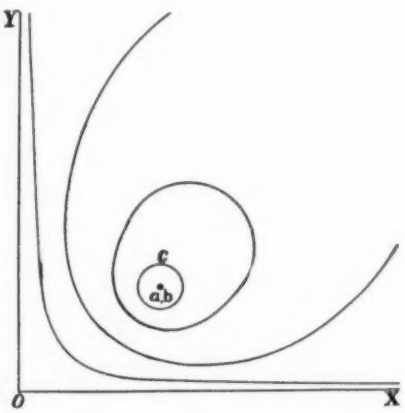


Fig. 1.

cross-section without any appreciable error if the radius r is small as compared to the distances a and b. Then the center of this circle can also be taken as the point (a, b). From (7) we have

$$u = \tfrac{1}{2} \log \frac{(x^2 - y^2 - a^2 + b^2)^2 + 4(xy + ab)^2}{(x^2 - y^2 - a^2 + b^2)^2 + 4(xy - ab)^2}. \tag{8}$$

Now since we are considering only the case where r is small as compared to a and b, we can find an expression for the potential at the surface of the wire by placing $y = b$ and $x = a - r$ in (8). Then

$$u = \tfrac{1}{2} \log \frac{(r - 2a)^2(r^2 + 4b^2)}{r^2[(r - 2a)^2 + 4b^2]}. \tag{9}$$

Now it can be easily proved that if u in equation (8) represents the potential at the surface of the conductor, the charge on it will be one half unit. Hence the capacity per unit length between the wire and the two planes is

$$C = \frac{\frac{1}{2}}{u - u_0},$$

where u_0 is the potential of the two plane conductors. But putting

$xy = 0$ in (8) gives $u_0 = 0$, and the capacity becomes

$$C = \frac{1}{2 \log \frac{\sqrt{(r-2a)^2(r^2+4b^2)}}{r\sqrt{(r-2a)^2+4b^2}}}. \tag{10}$$

If r is small enough that it may be neglected in comparison with a in the first power, the expression reduces to

$$C = \frac{1}{2 \log \frac{2ab}{r\sqrt{a^2+b^2}}}. \tag{11}$$

This same transformation may be used for finding the capacity between two horizontal parallel wires at equal distances above the earth, the one having a positive, the other an equal negative charge, and considering the surface of the earth as an infinite conducting plane at zero potential. In this case we have for the capacity

$$C = \frac{\frac{1}{2}}{u_1 - u_2},$$

where u_1 is the potential of the wire having the positive charge and u_2 is the potential of the other wire. To find the value of u_2, let $y = b$ and $x = -(a - r)$ and substitute in (8). Then

$$u_2 = \tfrac{1}{2} \log \frac{r^2[(r-2a)^2+4b^2]}{(r-2a)^2(r^2+4b^2)} \tag{12}$$

and hence $u_2 = -u_1$; which gives for the capacity

$$C = \frac{1}{\log \frac{\sqrt{(r-2a)^2(r^2+4b^2)}}{r\sqrt{(r-2a)^2+4b^2}}}, \tag{13}$$

or, neglecting r as compared to a and b, we have

$$C = \frac{1}{\log \frac{2ab}{r\sqrt{a^2+b^2}}}. \tag{14}$$

TWO WIRES PARALLEL TO THE EARTH AND ONE DIRECTLY ABOVE THE OTHER.

It was found that the equipotential lines and the lines of force about two wires parallel to the earth where one is directly above the other could be obtained by the transformation

$$w = \log \frac{(z^2 + a^2 + ad) - izd}{(z^2 + a^2 + ad) + izd}, \tag{15}$$

where a is the height of the lower wire and d is the distance between them. This is the case where the lower wire bears a positive charge and the upper wire a negative charge and the radius, r, of the wire is small compared to a and d. Then

$$u = \tfrac{1}{2} \log \frac{[x^2 + (y - a - d)^2][x^2 + (y + a)^2]}{[x^2 + (y + a + d)^2][x^2 + (y - a)^2]}. \tag{16}$$

The potential u_1 of the lower wire is found by putting $x = 0$ and $y = a + r$ in (16); hence

$$u_1 = \tfrac{1}{2} \log \frac{(d - r)^2(2a + r)^2}{(2a + d + r)^2 r^2}. \tag{17}$$

Likewise to obtain the potential of the upper wire, substitute $x = 0$ and $y = a + d + r$ in (16) which then becomes

$$u_2 = \tfrac{1}{2} \log \frac{r(2a + d + r)^2}{(2a + 2d + r)^2(d + r)^2}. \tag{18}$$

Whence the capacity for this system is

$$C = \frac{\tfrac{1}{2}}{2 \log \dfrac{\sqrt{(d - r)(2a + r)(2a + 2d + r)(d + r)}}{r(2a + d + r)}},$$

$$= \frac{1}{4 \log \dfrac{\sqrt{(d^2 - r^2)(2a + r)(2a + 2d + r)}}{r(2a + d + r)}}. \tag{19}$$

Neglecting the second power of r

$$C = \frac{1}{4 \log \dfrac{\sqrt{2d^2[(2a + r)(a + d) + ar]}}{r(2a + d)}}. \tag{20}$$

If we take a as infinite, we have simply the case of two parallel wires

for which the capacity is known to be

$$C = \frac{1}{4 \log \frac{l + \sqrt{l^2 - r^2}}{r}},$$

where $2l$ is the distance between them. Or neglecting r^2

$$C = \frac{1}{4 \log \frac{2l}{r}} = \frac{1}{4 \log \frac{d}{r}}, \tag{21}$$

where d is the distance between the wires. Now substituting a equal to infinity in (20) we obtain

$$C = \frac{1}{4 \log \frac{d}{r}}, \tag{22}$$

which agrees with the former result.

THREE PARALLEL WIRES ARRANGED SO AS TO BE AT THE CORNERS OF AN EQUILATERAL TRIANGLE.

In transmitting a three-phase alternating current by an overhead system, the wires are generally arranged so that they are at the corners of an equilateral triangle. We shall find the electrostatic capacity of such a system when the total charge on all the wires is zero. There will be no loss of generality if the charge on one of the wires is taken as one positive unit while that on each of the other two is taken as one half a negative unit. Again a very simple transformation can be found which will give the proper lines of force and equipotential. This transformation is

$$w = \log \frac{(z^2 - 4a^2) + 2azi}{(z^2 - 4a^2) - 4azi}, \tag{23}$$

where the distance between any two wires is $2a\sqrt{3}$. The equipotential lines will be found from the equation

$$u = \tfrac{1}{2} \log \frac{[(x - a\sqrt{3})^2 + (y + a)^2][(x + a\sqrt{3})^2 + (y + a)^2]}{[x^2 + (y - 2a)^2]^2}, \tag{24}$$

which is obtained directly fron (23) in the usual way.

In the above the origin has been taken at the intersection of the medians of the triangle and the wire bearing the positive charge has its center at the point (0, $2a$), while the other two wires have their centers

at the points $(a\sqrt{3}, -a)$ and $(-a\sqrt{3}, -a)$ respectively. Hence to find the potential of the first wire we shall place $x = 0$ and $y = 2a - r$ in (24) as for these small values of r the equipotential lines are approximately circles. Hence we obtain

$$u_1 = \tfrac{1}{2}\log\frac{[3a^2 + (3a - r)^2][3a^2 + (3a - r)^2]}{r^4}, \tag{25}$$

$$= \tfrac{1}{2}\log\frac{(12a^2 - 6ar + r^2)^2}{r^4}. \tag{26}$$

To find the potential of the wires bearing the negative charge, let $y = -(a - \frac{1}{2}r)$ and $x = \sqrt{3}(a - \frac{1}{2}r)$ and we have

$$u_2 = \tfrac{1}{2}\log\frac{r^2}{(12a^2 - 6ar + r^2)}. \tag{27}$$

Hence it is seen that $u_2 = -\frac{1}{2}u_1$; and the electrostatic capacity of the three wires, which is defined to be $1/u_1 - u_2$, becomes

$$C = \frac{2}{3u_1}, \tag{28}$$

$$= \frac{1}{3\log\left(\dfrac{\sqrt{12a^2 - 6ar + r^2}}{r}\right)}. \tag{29}$$

If these wires are considered parallel to the earth and the influence of the latter is considered the transformation takes the form

$$w = \log\frac{[(z - di)^2 - 3a^2][z + i(d + 3a)]}{[(z + di)^2 - 3a^2][z - i(d + 3a)]}, \tag{30}$$

where d is the height of the two lower wires above the surface of the earth. Then the expression for the potential is

$$u = \tfrac{1}{2}\log\frac{\{[x^2 - (y - d)^2 - 3a^2]^2 + 4x^2(y - d)^2\}\cdot[x^2 + (y + d + 3a)^2]^2}{\{[x^2 - (y + d)^2 - 3a^2]^2 + 4x^2(y + d)^2\}\cdot[x^2 + (y - d - 3a)^2]^2}. \tag{31}$$

The potential of the upper wire which bears the positive charge of one unit is found by placing $x = 0$ and $y = d + 3a - r$ in (31); which gives

$$u_1 = \tfrac{1}{2}\log\frac{[(3a - r)^2 + 3a^2]^2\cdot[2d + 6a - r]^4}{r^4[(2d + 3a - r)^2 + 3a^2]^2}. \tag{32}$$

As before it can be shown that

$$u_2 = -\tfrac{1}{2}u_1. \tag{33}$$

Hence the capacity of this system becomes

$$C = \frac{2}{3u_1},$$

$$= \frac{1}{3 \log \frac{(2d + 6a - r)}{r} \cdot \sqrt{\frac{(3a - r)^2 + 3a^2}{(2d + 3a - r)^2 + 3a^2}}}. \tag{34}$$

However for all practical cases r is very small as compared to d so that we may neglect r in (34) wherever it is added to d. This gives

$$C = \frac{1}{3 \log \left[\frac{(d + 3a)\sqrt{12a^2 - 6ar + r^2}}{r\sqrt{3a^2 + 3ad + d^2}} \right]}. \tag{35}$$

Placing d equal to infinite in the above equation will give the capacity of the three wires alone. This then is

$$C = \frac{1}{3 \log \left(\frac{\sqrt{12a^2 - 6ar + r^2}}{r} \right)}, \tag{36}$$

which is the same value found in equation (29).

It is also easily seen that the value of the capacity found in (35) is larger than that in (29), as was to be expected.

Images in a Cylinder.

Let us consider the transformation

$$w = \log \frac{R^2 - az}{R(z - a)}. \tag{37}$$

This gives for u

$$u = \log \frac{\sqrt{(R^2 - ax)^2 + a^2y^2}}{R\sqrt{(x - a)^2 + y^2}}. \tag{38}$$

For $u = 0$, we have

$$(R^2 - ax)^2 + a^2y^2 = R^2[(x - a)^2 + y^2],$$

whence

$$x^2 + y^2 = R^2. \tag{39}$$

That is, the zero potential surface is a cylinder and the cross-section in the xy-plane is a circle of radius R. Again from equation (38), we obtain

$$e^{2u} = \frac{(R^2 - ax)^2 + a^2y^2}{R^2[(x - a)^2 + y^2]}$$

and

$$x^2 + y^2 - 2x \cdot aR^2 \frac{e^{2u} - 1}{R^2e^{2u} - a^2} + R^2 \frac{a^2e^{2u} - R^2}{R^2e^{2u} - a^2} = 0,$$

or

$$\left(x - aR^2 \frac{e^{2u} - 1}{R^2e^{2u} - a^2}\right)^2 + y^2 = \left[\frac{e^uR(a^2 - R^2)}{R^2e^{2u} - a^2}\right]^2. \tag{40}$$

Hence the equipotential lines are all circles with their centers on the x-axis and at a distance from the origin given by

$$d = aR^2 \frac{e^{2u} - 1}{R^2e^{2u} - a^2}. \tag{41}$$

The radii are given by

$$r = \pm \frac{e^uR(a^2 - R^2)}{R^2e^{2u} - a^2}, \tag{42}$$

in which the positive sign is to be used for values of u greater than $u = \log(a/r)$ and the negative sign for values less than that. The

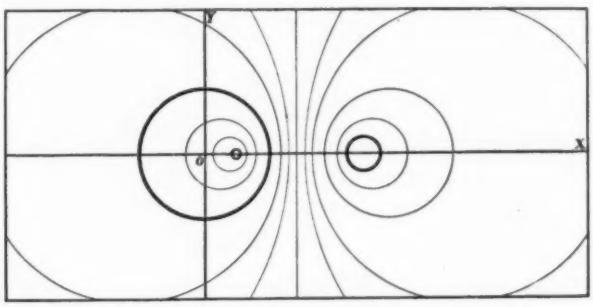


Fig. 2.

reason for this is quite obvious from Fig. 2. The radius becomes infinite for $u = \log(a/r)$ and the equipotential curve becomes the straight line

$$x = \frac{a^2 + R^2}{2a}.$$

Now let r_1 be the radius of a small wire bearing a charge of one half unit per unit length placed parallel to a large, earthed, cylindrical conductor of radius R and let d be the distance between the centers of the two. Then we can replace the cylinder by another wire bearing a negative charge of one half unit without any change in the electrostatic field. To find the radius r_2 and the position of this latter wire replace u by $-u$ in equations (41) and (42). It will be seen that r_2 is less than r_1 as might

be expected. For

$$r_1 = \frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2}, \quad r_2 = \frac{e^u R(a^2 - R^2)}{a^2 e^{2u} - R^2}. \tag{43}$$

Also the positions of the centers of the wires will be given by

$$d_1 = aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2}, \quad d_2 = aR^2 \frac{e^{2u} - 1}{a^2 e^{2u} - R^2}. \tag{44}$$

It is to be understood that the above reasoning only holds for the case where very long and large cylinders are considered. It can readily be shown that the method of images does not apply rigorously to the case of two long small parallel wires at any considerable distance. That is, as suggested earlier in this paper, the electric force in the direction of the z-axis must be so small that it may be neglected. However in this as in the general theory of the logarithmic potential, the very long wires must be considered as finite in length when applying the test of zero potential at an infinite distance.

Now equation (38) may be used for finding the capacity between a wire bearing a charge and an earthed wire near it. First consider the case where the small wire is external to the earthed cylinder. The first of equations (43) gives

$$r_1 = \frac{e^u R(a^2 - R^2)}{R^2 e^{2u} - a^2},$$

from which we find

$$a^2 = \frac{R^2(e^u R + r_1 e^{2u})}{e^u R + r_1}. \tag{46}$$

From the first of equations (45)

$$d_1 = aR^2 \frac{e^{2u} - 1}{R^2 e^{2u} - a^2},$$

whence

$$d_1^2 = \frac{(R + r_1 e^u)(e^u R + r_1)}{e^u}.$$

Solving for e^u,

$$e^u = \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2 r_1^2}}{2Rr_1}, \tag{47}$$

and

$$u = \log \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2 r_1^2}}{2Rr_1}. \tag{48}$$

Now it is easily proved that the charge on the wire is one half a unit, so that the capacity of this system is

$$C_1 = \frac{1}{2 \log \frac{(d_1^2 - r_1^2 - R^2) + \sqrt{(d_1^2 - r_1^2 - R^2)^2 - 4R^2 r_1^2}}{2Rr_1}}. \quad (49)$$

Likewise the capacity between the internal wire and the cylinder may be found from the second of the equations (44) and (45) respectively. This gives

$$C_2 = \frac{1}{2 \log \frac{(R^2 + r_2^2 - d_2^2) + \sqrt{(R^2 + r_2^2 - d_2^2)^2 - 4r_2^2 R^2}}{2r_2 R}}. \quad (50)$$

To prove that the above expressions have the proper form, place $d = 1 + R$ and let R become infinite; that is, let the earthed cylinder become an infinite earthed plane. This gives

$$C = \frac{1}{2 \log \frac{l + \sqrt{l^2 - r^2}}{r}},$$

which is the identical expression previously obtained for a wire parallel to an infinite earthed plane.

An expression for the surface distribution over the earthed cylinder is very easily obtained. From equation (37) we obtain

$$\frac{dw}{dz} = \frac{R^2 - a^2}{az^2 - z(R^2 + a^2) + aR^2}.$$

Taking the absolute value and simplifying

$$\left|\frac{dw}{dz}\right| = \frac{R^2 - a^2}{\sqrt{[a(x^2 - y^2 + R^2) - x(R^2 + a^2)]^2 + [y(2ax - R^2 - a^2]^2}}.$$

Transforming to polar coördinates

$$\left|\frac{dw}{dz}\right| = \frac{R^2 - a^2}{R(R^2 + a^2 - 2aR \cos \varphi)}.$$

Hence

$$\sigma = \frac{1}{4\pi} \frac{R^2 - a^2}{R(R^2 + a^2 - 2aR \cos \varphi)} \quad (51)$$

and a can be calculated for any particular system from equation (46). It is seen that the maximum value of σ is for $\varphi = 0$ and the minimum value for $\varphi = \pi$.

Two Core Cable.

We shall next consider the transformation

$$w = \log \frac{(z+a)(z-b)}{(z-a)(z+b)}.$$

This gives

$$u = \tfrac{1}{2} \log \frac{[(x+a)^2+y^2][(x-b)^2+y^2]}{[(x-a)^2+y^2][(x+b)^2+y^2]},$$

from which the zero potential curves are found to be

$$x = 0$$

and

$$x^2 + y^2 = ab.$$

The latter suggests the possibility of using the transformation for the cases where wires are encased in an earthed sheath of circular cross section, and where the total charge on the cable at any instant is zero.

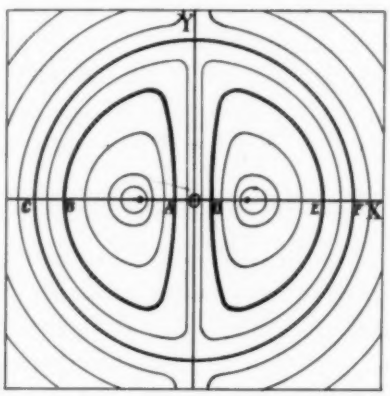


Fig. 3.

The following results are for air as the dielectric, but of course the results can be altered to fit the case for any dielectric.

If the radius of this sheath is R, we have

$$ab = R^2.$$

Replacing b in the original equation by (R^2/a) gives

$$w = \log \frac{(z+a)(az-R^2)}{(z-a)(az+R^2)}, \tag{52}$$

and for the potential

$$u = \tfrac{1}{2} \log \frac{[(x+a)^2+y^2][(ax-R^2)^2+a^2y^2]}{[(x-a)^2+y^2][(ax+R^2)^2+a^2y^2]}. \tag{53}$$

The equation of the equipotential lines becomes

$$e^{2u}[(x-a)^2+y^2][(ax+R^2)^2+a^2y^2] = [(x+a)^2+y^2][(ax-R^2)^2+a^2y^2].$$

These curves are shown in Fig. 3. In many cables,[1] used especially in Europe, a bundle of wires is so arranged that the contour exactly coincides with the heavy curves AB and DE shown in Fig. 3.

If $a = R/3$, we shall have $AD = EF$. This is the most usual condition. Substituting $R/3$ for a in (53), we obtain

$$u = \tfrac{1}{2}\log\frac{\left[\left(x+\frac{R}{3}\right)^2+y^2\right]\left[\left(\frac{Rx}{3}-R^2\right)^2+\frac{R^2y^2}{9}\right]}{\left[\left(x-\frac{R}{3}\right)^2+y^2\right]\left[\left(\frac{Rx}{3}+R^2\right)^2+\frac{R^2y^2}{9}\right]}. \tag{54}$$

Now let $AD = EF = 2d$; then for $y = 0$, $x = d$ we obtain the potential of the bundle of wires DE, which is

$$u_1 = \log\frac{(3d+R)(d-3R)}{(3d-R)(d+3R)}, \tag{55}$$

$$= \log\frac{3R^2+8Rd-3d^2}{3R^2-8Rd-3d^2}.$$

Likewise the potential of the bundle of wires AB will be obtained by placing $y = 0$ and $x = -d$. Hence

$$u_2 = \log\frac{(3d-R)(d+3R)}{(3d+R)(d-3R)}.$$

Hence $u_2 = -u_1$ and the capacity per unit length between the two bundles of wires, one positively the other negatively charged, surrounded by an earthed cylinder of circular cross-section is

$$C = \frac{\frac{1}{2}}{2u_1},$$

$$= \frac{1}{4\log\left[\frac{3(R^2-d^2)+8Rd}{3(R^2-d^2)-8Rd}\right]}. \tag{56}$$

Another case of interest is that in which two small wires are encased in the circular sheath. As is seen in Fig. 3, the smaller equipotential lines approximate to circles. Making this approximation, we shall

[1] See Russell's Alternating Currents, Vol. I, Chapters IV. and V.

obtain the potential of the wire bearing the positive charge by placing $y = 0$ and $x = a - 2r/3$ in (53), which gives the case for $a = R/3$. Hence

$$u = \log \frac{(2r - 6a)(3a^2 - 2ar - 3R^2)}{2r(3a^2 - 2ar + 3R^2)}. \tag{57}$$

Then placing $a = R/3$, we have

$$u = \log \frac{(R - r)(4R - r)}{r(5R - r)}.$$

And the capacity between the two wires within the earthed sheath is

$$C = \frac{1}{4 \log \frac{(R - r)(4R - r)}{r(5R - r)}}. \tag{58}$$

It is to be understood that this latter form is an approximation that can be used only for small wires so placed that the distance from the surface of one wire to the surface of the other is equal to the distance from the inner surface of the metal sheath to the surface of the nearest wire. However the method can be adapted to other conditions by taking the proper relations between a, r, and R.

Three Core Cables.

A transformation was found which would give equipotential lines which very nearly coincide with those in the clove-leaf type of three-core cable. A diagram of this cable is shown in Russell's Alternating Currents, Vol. I. This transformation is

$$w = \log \frac{[(z^2 - R^2) + Rzi] \cdot (z - 4Ri)^2}{[(z^2 - 16R^2) + 4Rzi](z - Ri)^2}. \tag{59}$$

where the distance between the surfaces of the bundles of wires is equal to the distance from the inner surface of the enclosing metal sheath to the surface of one bundle of wires. The charge on one of the conductors is one positive unit, while each of the other two has a negative charge of one half unit. Then the equation of an equipotential line becomes

$$u = \tfrac{1}{2} \log \frac{[(x^2-y^2-R^2-Ry)^2+x^2(2y+R)^2][x^2+(y-4R^2]^2}{[(x^2-y^2-16R^2-4Ry)^2+x^2(2y+4R)^2] \cdot [x^2+(y-R)^2]^2}. \tag{60}$$

Now if d is the distance from the center of the circular cylinder to the

outer surfaces of the bundles of wires, an expression for the potential of the wire bearing the positive charge will be obtained by substituting $x = 0$ and $y = d$ in (60). This gives

$$u_1 = \log \frac{(R^2 + d^2 + Rd)(4R - d)^2}{(16R^2 + d^2 + 4Rd)(R - d)^2},$$

$$= 2 \log \left[\frac{(4R - d)\sqrt{(R + d)^2 - Rd}}{(R - d)\sqrt{(4R + d)^2 - 4Rd}} \right]. \tag{61}$$

Now it can be proved that the potential of the conductors bearing the negative charge is

$$u_2 = -\tfrac{1}{2}u_1.$$

Hence the capacity of the system is

$$C = \frac{1}{3 \log \left(\frac{4R - d}{R - d} \cdot \sqrt{\frac{(R + d)^2 - Rd}{(4R + d)^2 - 4Rd}} \right)}. \tag{62}$$

If the conductors are small wires of circular cross-section, and radius r, a close approximation to the value of the capacity will be obtained by substituting $d = \frac{1}{2}R + r$ in (62). For this condition then

$$C = \frac{1}{3 \log \left(\frac{7R - 2r}{2R - 4r} \cdot \sqrt{\frac{7R^2 + 8Rr + 4r^2}{18R^2 + 5Rr + r^2}} \right)} \tag{63}$$

It may seem at first that too many approximations have been made in this paper, but a closer examination will show that the results obtained by using the formulæ derived will, for nearly all practical cases, be more accurate than the measurements from which the calculations are made. In the cases of the cables if the wires have the shape of the equipotential lines, the results will be exact.

In conclusion I wish to thank Dr. J. Kunz and Professor E. J. Townend for their many helpful suggestions during the investigation of the above problems.

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August, 1912.

SOME REMARKABLE CASES OF RESONANCE.[1]

By C. V. Raman.

THE general principle of resonance is that a periodic force acting on an oscillatory system may set up and maintain a large amplitude of vibration when the periods of the force and of the system are approximately equal, even though in other cases the amplitude might be so small as to be negligible. In the present paper I propose to discuss some remarkable cases which form apparent exceptions to this law of equality of periods, that is, in which we have marked resonance when the periods of the impressed force and of the system do *not* stand to each other in a relation of approximate equality.

The first of these is the well-known case of double frequency, the theory of which was first discussed by Lord Rayleigh.[2] In a note published in the Physical Review for March, 1911, I promised a fuller discussion of the modification in this theory necessary to fit the results with those actually observed in experiment. I now proceed to fulfil this promise and the delay that has occurred in doing so is I feel a matter for regret. Lord Rayleigh starts with the following as his equation of motion:

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt)U = 0, \tag{1}$$

and assuming that U, the displacement at any instant during steady motion, can be represented by an expression of the form

$$A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt, \tag{2}$$

proceeds to find the conditions that must be satisfied for the assumed steady motion to be possible. This he does by substituting (2) for U in the left-hand side of equation (1) and equating to zero the coefficients of sin pt, cos pt, etc. The conditions for the possibility of steady motion thus obtained are

$$\frac{B_1}{A_1} = \frac{\sqrt{(\alpha - kp)}}{\sqrt{(\alpha + kp)}} = \tan e, \tag{3}$$

$$(n^2 - p^2)^2 = \alpha^2 - k^2p^2. \tag{4}$$

By a trigonometrical transformation equation (3) can be written in the

[1] Preliminary notes on this subject appeared in Nature, December 9, 1909, and February 10, 1910, and in the Physical Review, March, 1911.

[2] Phil. Mag., April, 1883, and August, 1887, and Theory of Sound, Art. 68 (*b*).

form

$$kp = \alpha \cos 2e. \tag{5}$$

It appears from these equations that the phase of the oscillation maintained, *i. e.*, e is independent of the amplitude, and that the latter quantity is indeterminate.

I attempted to verify the phase-relation given by equations (4) and (5) experimentally in the following way: The oscillating system used was a stretched string and this was maintained in motion by periodically varying its tension in the manner of Melde's experiment. Since the periodic change of double frequency in the tension of the string is imposed by the tuning-fork, the motion of the prong corresponds to the term $-2\alpha \sin 2pt$ in equation (1) and the transverse vibration of the string to the expression $(A_1^2 + B_1^2)^{1/2} \sin (pt + e)$ if the small terms in A_3, B_3, etc., are neglected. The experimental problem therefore reduces itself to the determination of the phase-relation between the motion of the string and the vibration of the prong of the exciting tuning-fork. This can be attacked by two distinct methods. The first is

Mechanical Composition of the two motions. This is automatically effected and needs no special experimental device. For, the motion of the prong is longitudinal to the string and any point on the string near the end attached to the prong or near any other intermediate node of the oscillation has *two* rectangular motions: the first longitudinal to the string and having the same frequency as the vibration of the fork; and the second which is the general transverse oscillation of the string. The resulting motion is in a Lissajous figure and this is readily observed by attaching a fragment of a silvered bead to a point on the string near the fork.

Optical Composition of the two motions furnishes a second method and this is undoubtedly the more elegant of the two. It can be effected in the following way: A small mirror is attached normally to the extremity of the prong of the fork. The plane of the oscillation of this mirror is perpendicular to that of the vibration of the string, and a point on the latter is brightly illuminated by a transverse sheet of light from a lantern or from a cylindrical lens. When the string is in oscillation the illuminated point appears drawn out into a straight line, and this is viewed by reflection first at a fixed mirror and then at the mirror attached to the vibrating prong. The illuminated point is then seen to describe a Lissajous figure which is compounded of the motions of the string and the tuning-fork.

Observing by either of the methods described above, the relation

between the phases of the transverse oscillation of the string and of the motion of the prong of the tuning-fork can be closely studied, and some remarkable phenomena are noticed in this way. The principal point observed is that the phase relation is not independent of the amplitude maintained. This is best shown by using a bowed fork and starting with a large amplitude of motion and then allowing the motion to die away. The initial curve of motion and the changes in it as the motion dies away both depend on the tension of the string. When this is in excess of that required for the most vigorous maintenance, the curve is a parabolic arc convex to the fork and remains as such when the motion dies away. With a smaller tension adjusted so that the free period of the string is exactly double that of the fork, the initial curve with a large amplitude of motion is still approximately a parabolic arc convex to the fork. The damping of the motion is now more rapid and the curve reduces to an 8-shaped figure when the amplitude is very small. For the most vigorous maintenance a still smaller tension is necessary and the initial curve with a large motion is still convex to the fork, but it will now be noticed that when the amplitude falls to a very small quantity the curve passes through the 8-figure stage and tends to become *concave* to the fork. The most remarkable changes are observed when the tension is smaller still. The damping here is very large and a steady motion is only possible when the amplitude exceeds a certain minimum value. At this stage the string very rapidly comes to rest and in the final stage the curve of motion becomes a parabolic arc *concave* to the fork with a very small amplitude. For a satisfactory explanation of these phenomena it is necessary to start with a modified equation of motion which takes into account the variations of tension which exist in *free* oscillations of sensible amplitude and are proportional to the square of the motion. The equation of motion thus completed is

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt + \beta U^2) \cdot U = 0. \tag{6}$$

Substituting expression (2) for U in the left-hand side of the above given equation and putting the coefficients of $\sin pt$, $\cos pt$, etc., equal to zero, the conditions for the possibility of steady motion reduce to the form

$$kp = \alpha \cos 2e \tag{7}$$

and

$$(n^2 - p^2 + F)^2 = \alpha^2 - k^2p^2, \tag{8}$$

where

$$F = \frac{3\beta}{4}(A_1^2 + B_1^2).$$

Equation (8) determines the amplitude of the motion and (7) its phase. It is evident at once that with given values for α and kp the amplitude of the motion is greatest when n is smallest and that there can be no maintenance if $\alpha < kp$. We therefore get the apparently paradoxical conclusion (which is amply verified by experiment) that the maintenance is *not* the most vigorous when the free period (for small amplitudes) of the string is double that of the fork. Another interesting inference which is confirmed by experiment is that when $n < p$ and $(n^2 - p^2)^2 > (\alpha^2 - k^2p^2)$ maintenance is impossible unless F, *i. e.*, also the amplitude, has a definite minimum value.

Equation (7) shows that as α is increased e, the phase of the oscillation, alters continuously. The influence of F, *i. e.*, of the amplitude of the motion on e its phase can readily be traced from equation (8). When $n > p$, for a larger value of F we must have a large value of α and $\cos 2e$ tends more and more to assume a zero value. Writing equation (7) in the form

$$\tan e = \frac{\alpha - kp}{n^2 - p^2 + F} \tag{9}$$

it is evident that when $n > p$ and F is large, e is positive and approaches to the value $\pi/4$. This agrees with the experimental result. When $n < p$, e may be positive or negative according as F is greater or less than $(p^2 - n^2)$ and the alteration of the phase of the motion with the amplitude is most conspicuous, and this is in agreement with observation. In the extreme lower limit e tends to the value $-\pi/4$, and the curve of motion is a parabolic arc *concave* to the fork. In other words when the prong is at its extreme outward string, the string is also at its extreme outward swing, a seemingly paradoxical result not in accordance with the ordinarily received ideas of the experiment. Equation (1) may be written in the form

$$\ddot{U} + k\dot{U} + n^2U = 2\alpha U \sin 2pt. \tag{10}$$

The right-hand side of this equation may be regarded as the *impressed* part of the restoring force acting on the system, and this is a very useful way of regarding the matter. Putting $U = P \sin (pt + e)$ to a first approximation, we can find the conditions that must exist for steady motion *directly* by equating the work done by the force represented by the right-hand side term of equation (10) to the energy dissipated in an equal time by the friction term on the left. The relation thus obtained is

$$kp = \alpha \cos 2e,$$

which is identical with equation (7) obtained from the complete analysis. It is observed that the right-hand side of equation (10) is

$$2\alpha P \sin 2pt \sin (pt + e)$$

and this may be written as

$$\alpha P (\cos \overline{pt - e} - \cos \overline{3pt + e}).$$

The second term within the brackets is ineffective so far as the maintenance of the motion $P \sin (pt + e)$ is concerned. We may therefore leave it out and write equation (10) in the form

$$\ddot{U} + k\dot{U} + n^2 U = \alpha P \cos (pt - e). \qquad (11)$$

Written in this way it is evident that a large motion must ensue if $p = n$ and that we have here merely an example of the general principle of resonance.

PART II.

I now proceed to consider some other exceedingly interesting cases of resonance under the action of forces similar in character to that in the case of double frequency considered above, but having other frequency relations to the system influenced. My experiments show that resonance may occur in the following cases of the kind:

(1) When the period of the force is $\frac{1}{2}$ that of the system.
(2) When the period of the force is $\frac{2}{2}$ times that of system.
(3) When the period of the force is $\frac{3}{2}$ times that of the system.
(4) When the period of the force is $\frac{4}{2}$ times that of the system.
(5) When the period of the force is $\frac{5}{2}$ times that of the system.
(6) When the period of the force is $\frac{6}{2}$ times that of the system.

And so on.

The most remarkable instances of these cases of resonance are furnished by a stretched string under the action of a periodically varying tension. To observe them, all that is required is that the tension of the string should be gradually increased till its free period in the fundamental mode stands in the desired relation to the period of the tuning-fork which imposes the variable tension. It will then be found that a vigorous oscillation is maintained. Figs. 1, 2, 3, 4 and 5 are photographs of stretched strings maintained in the first five types of motion respectively under the action of an electrically maintained tuning fork varying the tension periodically.

The actual frequency and the phase of the maintained motion in each of these cases can be determined by observation of the corresponding Lissajous figures, using the mechanical or optical method of composition described above for the first of these cases. The detailed results I must reserve for a future paper. One good way of studying these types of motion is to illuminate one point on the vibrating string by a transverse sheet of light from a lantern or from a cylindrical lens and to observe the line of light so produced in a revolving mirror. But the best method of all for recording the motion photographically is that by which I obtained Figs. 6, 7, 8, 8 (*a*), 9 and 10 published herewith and which I now proceed to describe.

Figs. 6, 7, 8, 9 and 10 refer respectively to the first five types of motion as shown in Figs. 1, 2, 3, 4 and 5. It will be observed that each of them shows two curves. The white curve in the black ground is a record of the motion of the tuning-fork, and the other curve which is black on a white ground is a record of the motion of a point on the string maintained in vibration. These records were obtained on a moving photographic plate in the following manner. One source of light was a horizontal slit, and the other was a vertical slit placed behind the oscillating string. Both were illuminated by sunlight and had collimating lenses in front of them. The light from the former fell upon a small mirror attached to the prong of the vibrating fork and after reflection fell upon the lens (having an aperture of 4 cm. diameter) of a roughly constructed camera. The light from the vertical slit behind the vibrating string was also reflected into the camera by a fixed mirror. In the focal plane of the camera was placed a brass plate with a vertical slit cut in it. The images of the horizontal and vertical slits fell, one immediately above the other, on the slit in the plate. Only a very small length of the former, *i. e.*, practically only a point of light was allowed to fall upon the photographic plate. The dark slide which held this was moved uniformly by hand in horizontal grooves behind the slit in the focal plane of the camera, while the fork and the string were in oscillation.

Figs. 1 and 6 represent the well-known case in which the string makes one oscillation for every two oscillations of the fork. This is evident in the photograph.

Figs. 2 and 7 represent the next type in which the variable tension maintains an oscillation of the same frequency as its own. It will be noticed that the curvature of one of the extreme positions of the string is somewhat greater than that of the other and that the mid-point of the oscillation is somewhat displaced to one side of the middle-point of the vertical slit at which the string was set when at rest. The inference

from this fact of observation is that the transverse motion of each point on the string is represented by an expression of the form

$$P \sin (2pt + e) + Q, \tag{12}$$

where Q is a constant. A motion of this type is only possible under a variable spring. For, the restoring forces acting on an element of the string at the two unequally curved extremes of its swing cannot themselves be equal and opposite (the condition necessary for a simple harmonic oscillation) unless the tensions of the string at the two extreme positions are unequal. In fact the second constant term Q in the motion is introduced under the action of the variable spring, and its importance will become evident as we proceed.

Fig. 3 and 8 and 8 (*a*) represent the third type of motion in which the string makes *three* swings for every *two* swings of the fork. But it is evident from Figs. 8 and 8 (*a*) that the successive swings on opposite sides are not all equal in amplitude and the influence of this is also perceptible in Fig. 3, having given rise to the appearance of two extra strings, which represent really the turning points of the motion. The vibration curve shown in Figs. 8 and 8 (*a*) can be represented by an expression of the form

$$P \sin (3pt + e) + Q \sin (pt + e'). \tag{13}$$

The alternate increase and decrease of the amplitude of the motion of the string is evidently due to the action of the varying tension and the term $Q \sin (pt + e')$ in the motion which superposed on the first reproduces this waning and waxing effect, plays a very important part in the maintenance of the motion, as we shall see later on. Figs. 4 and 9 represent the fourth type of motion in which the string makes *four* swings for every *two* swings of the fork. As before the waning and waxing of the motion under the action of the variable spring is evident in the photographs and the observed motion of the points on the string is of the type

$$P \sin (4pt + e) + Q \sin (2pt + e'). \tag{14}$$

Here as before we shall see that the second term which is introduced under the action of the variable spring plays a very important part in the maintenance of the motion.

Figs. 5 and 10 represent the fifth type of motion in which the string makes five swings for every two swings of the fork. The periodic increase and decrease in the amplitude of the motion is also evident. The vibration-curve may be represented by

$$P \sin (5pt + e) + Q \sin (3pt + e'), \tag{15}$$

the second term being due to the action of the variable tension.

In the general case therefore we may assume the maintained motion to be of the form

$$P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2}pt + e'). \qquad (16)$$

The equation of motion under a variable spring may be written as

$$\ddot{U} + k\dot{U} + n^2U = 2\alpha U \sin 2pt.$$

(see equation (10) above).

If now we substitute (16) for U, the right-hand side of equation (10) gives us what we may regard as the impressed part of the restoring force at any instant. It may be written as

$$\alpha P[\cos (\overline{\gamma - 2}pt + e) - \cos (\overline{\gamma + 2}pt + e)] + \alpha Q [\cos (\overline{\gamma - 4}pt + e') - \cos (\gamma pt + e')]. \qquad (17)$$

The work done by this force in a period of time t embracing any number of complete cycles of the variable spring is found on integration to be equal to $PQ\alpha pt \cos (\pi + e - e')$ if $\gamma > 2$ or to $2PQ\alpha pt \sin e \sin (e' - \pi)$ if $\gamma = 2$. The surplus of energy thus made available may be sufficient to counteract the loss by dissipation in the same time, *i. e.*, to maintain the motion.

It is not difficult to make out from equations (10) and (17) that these apparently anomalous cases in reality form illustrations of the general principle of equality of periods required for resonance. For, we get a large motion when $n = \gamma p$ in the general case, and the reason for this is evident at once if we neglect the first three terms in (17) as ineffective and write equation (10) as under

$$\ddot{U} + k\dot{U} + n^2U = -\alpha Q \cos (\gamma pt + e'). \qquad (18)$$

We started on the assumption that

$$U = P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2}pt + e').$$

Equation (18) shows that if we had neglected the second term (coefficent Q) we should have been unable to account for the resonance effect observed. Probably for a more complete discussion it would be necessary to take three terms thus:

$$U = P \sin (\gamma pt + e) + Q \sin (\overline{\gamma - 2}pt + e') + R \sin (\overline{\gamma + 2}pt + e_1).$$

Experiment shows however that the third term (with coefficient R) is relatively unimportant and the treatment given above may therefore be taken as a fairly close first approximation. It will be noticed from Figs. 7, 8, 8 (*a*), 9 and 10 that the epochs of maximum amplitude in each case pretty closely correspond to those of minimum tension and vice versa. This is exactly what is to be expected on general considerations.

Some very curious phenomena are observed when the vibration of the string in each of the cases described above is observed through a stroboscopic disk. These and other matters I hope to detail in a future paper.

PART III.

In part II. the equations of motion discussed are throughout those of a body having one degree of freedom. This was sufficient for the purpose of elucidating the leading features of each type of motion considered. But it must not be overlooked that the systems dealt with, *i. e.*, stretched strings, have more than one normal mode of motion and this fact leads to certain exceedingly interesting complications. The phenomena observed under this heading fall into two distinct classes which I shall discuss separately.

The first class of phenomena I have designated "transitional types of oscillation." Their existence may be explained somewhat as follows: Take the case of a system maintained in one of its natural modes of vibration by periodic forces of double frequency. It is evident that the actual period of vibration would be *exactly* double the period of the acting force but the *free* period of vibration in the particular mode may differ slightly from the *forced* period of vibration. The range and extent of the permissible difference between the two is a function of the magnitude of the periodic force acting on the system. Assume now that the system has another natural mode of vibration whose frequency for free oscillations is not very far removed from that of the first and that the magnitude and frequency of the periodic force acting on the system is such that the ranges of the two natural modes of vibration for maintenance by forces of double frequency partly overlap and the force actually at work falls within the overlapping part. It is evident that in such a case the system would vibrate with a frequency equal to exactly half that of the acting force, but the *mode* of vibration would not be either of the natural modes but something intermediate between the two. These "transitional" types or modes of motion possess special experimental interest in the case of stretched strings as they can be readily observed and studied. It is not at all difficult for instance to maintain a "transition" mode of oscillation intermediate between the ordinary modes with three and four ventral segments respectively, by suitably adjusting the tension and varying it periodically by the aid of a tuning-fork. The frequency of the motion would everywhere be exactly half that of the fork and the motion at each point strictly "simple harmonic," but there would be no "nodes" or points of rest visible. Such a type of motion presents a very remarkable appearance when examined under

intermittent illumination of periodicity slightly different from that of the tuning-fork. The two intersecting positions of the string seen undergo a periodic cycle of changes enclosing alternately three and four ventral segments.

The phenomena observed in the experiment described above can be explained on the supposition that the displacement at each point can be represented by the equation

$$y = a \sin \frac{3\pi x}{l} \sin (pt + e) + b \sin \frac{4\pi x}{l} \sin (pt + e'). \qquad (19)$$

Equation (19) suggests that the phase of the motion is not the same at all points of the string. In fact working by the optical and mechanical methods described in the first part of this paper I observed very remarkable variations of phase over the length of the string. It appeared that in some cases e and e' differed by as much as $\pi/2$.

Of course we should get "transitional types" of oscillation with the vibrations of higher frequencies maintained by periodic forces which were discussed in part II. of this paper, but they are not so marked as in the case of double frequency since the frequency-ranges become smaller as we go up the series.

The second class of phenomena observed cannot be fully discussed within the limits of the present paper and I shall have to content myself with briefly indicating their nature. In part II. of this paper I showed that a variable tension or "spring" may under suitable circumstances maintain an oscillation of a frequency standing in any one of a series of ratios to its own frequency. If the system which is subject to the variable "spring" or tension has itself a series of natural modes or frequencies, it would evidently be possible for two or more modes of vibration to be set up simultaneously with the respective frequencies and we would find a "simple harmonic" variation of tension maintaining a *compound* vibration. The special interest of this in the case of stretched strings consists of the fact that the natural frequencies of the system themselves form a harmonic series, and we may also have oscillations set up independently by one and the same force in rectangular planes and the compound character of the motion would be rendered visible by the curved paths of points on the string. These curves would in fact be identical with or analogous to the respective Lissajous figures and I hope with a future paper to publish several photographs which I have taken of compound vibrations maintained in this manner by a single tuning-fork. Two of these will be found published with my note in Nature, Feb. 10, 1910.

FIG. 1.

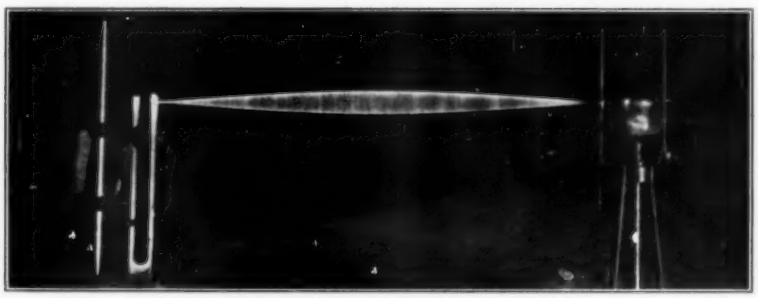

FIG. 2.

FIG. 3.

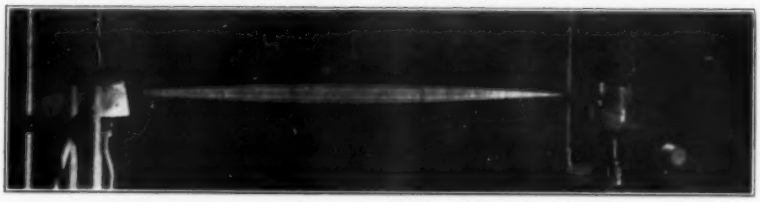

FIG. 4.

FIG. 5.

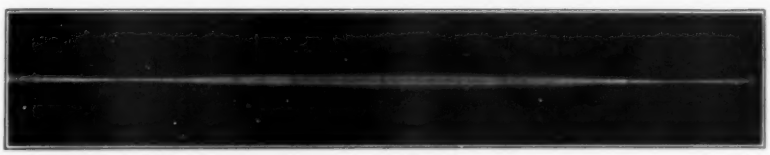

FIG. 6.

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FIG. 7.

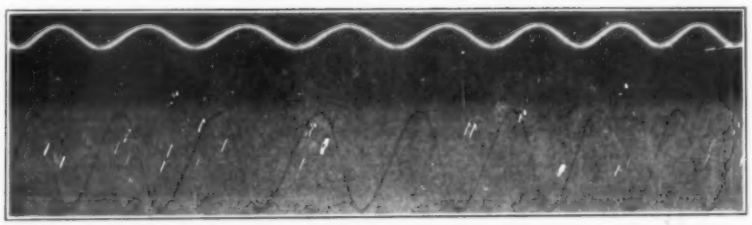


FIG. 8.

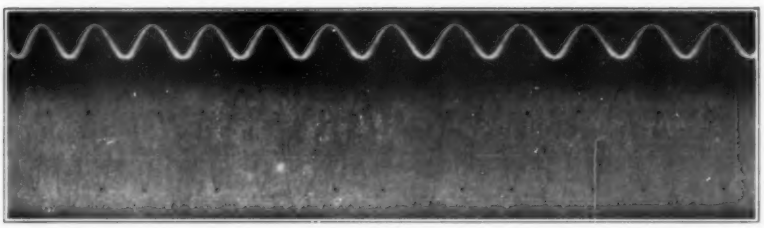


FIG. 8a.

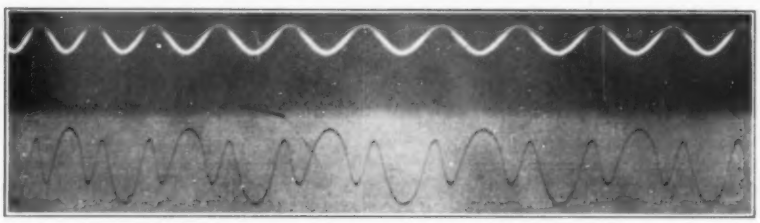


FIG. 9.

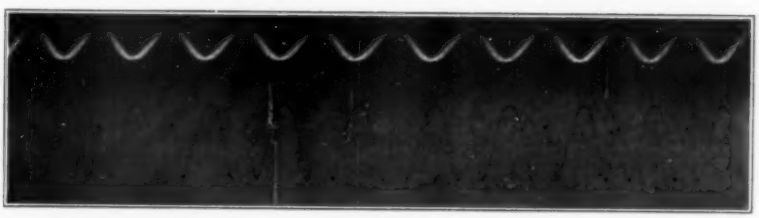
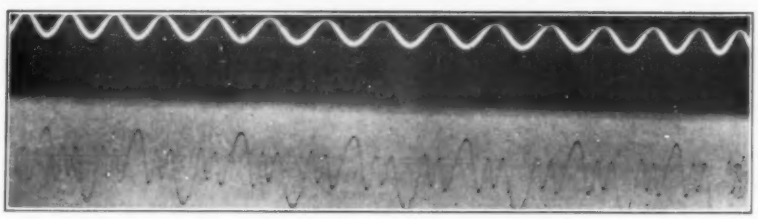


FIG. 10.



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THE POTENTIAL AND ELECTROSTATIC FORCE IN THE FIELD OF TWO METAL SPHERICAL ELECTRODES.

By Geo. R. Dean.

I. The Potential at any Point.

TWO conducting spheres are charged to potentials V_1 and V_2. The ratio of the distance of any point from the two inverse points of the spheres being denoted by e^{η} and the angle between them by ξ, the potential at the point (η, ξ) is

$$V_1\{\sqrt{2(\cosh\eta - \cos\xi)}\}\sum_0^\infty \frac{\sinh(m+\frac{1}{2})(\beta+\eta)}{\sinh(m+\frac{1}{2})(\alpha+\beta)} P_m(\cos\xi)\, e^{-(m+1/2)\alpha}$$

$$+ V_2\{\sqrt{2(\cosh\eta - \cos\xi)}\}\sum_0^\infty \frac{\sinh(m+\frac{1}{2})(\alpha-\eta)}{\sinh(m+\frac{1}{2})(\alpha+\beta)} P_m(\cos\xi)\, e^{-(m+1/2)\beta}, \quad (1)$$

where $P_m(\cos\xi)$ is a zonal harmonic, and $\eta = \alpha$ on the surface of the sphere whose potential is V_1 and $\eta = -\beta$ on the surface of the sphere whose potential is V_2.

Before giving the details of the derivation of (1) it will make the matter more interesting to the reader to give an outline of the process.

The function (1) is a solution of Laplace's equation, which in the case of symmetry with respect to an axis, as in the case of the spheres, is

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{z}\frac{\partial V}{\partial z} = 0, \quad (2)$$

r being measured along the axis, and z normal to it.

The first step is to transform (2) to new independent variables defined by the equations

$$n = \tfrac{1}{2}\log\left[\frac{z^2 + (c+r)^2}{z^2 + (c-r)^2}\right], \quad \xi = \tan^{-1}\frac{2cz}{r^2 + z^2 - c^2}, \quad (3)$$

taking the origin at the middle point of the line between the limiting or inverse points. The position of these points is determined in what follows.

The transformed equation is

$$\frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{\sinh\eta}{\cosh\eta - \cos\xi}\frac{\partial V}{\partial \eta} + \frac{\cosh\eta\cos\xi - 1}{\sin\xi(\cosh\eta - \cos\xi)}\frac{\partial V}{\partial \xi} = 0. \quad (4)$$

Equation (4) is transformed by the substitution $V = U\sqrt{2(\cosh\eta - \cos\xi)}$ into

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2} + \cos\xi \frac{\partial U}{\partial \xi} - \frac{U}{4} = 0. \tag{5}$$

Substituting $U = PQ$, where P is a function of η alone and Q a function of ξ alone, we get the ordinary equations

$$\frac{d^2P}{d\eta^2} - (m + \tfrac{1}{2})^2 P = 0, \qquad \frac{d^2Q}{d\xi^2} + \cot\xi \frac{dQ}{d\xi} + m(m+1)Q = 0. \tag{6}$$

The particular solutions of (6) are used in the manner described below to build up a solution to fit the given conditions, which are that $V = V_1$ when $\eta = \alpha$, and $V = V_2$ when $\eta = -\beta$.

The Inverse Points of Two Circles.—By a well-known theorem in geometry, the locus of a point which moves so that the ratio of its distance from two fixed points is constant, is a circle, and if C be the center of the circle and A and B the two fixed points, A, B and C are collinear and

$$CA \times CB = a^2.$$

The points A and B are called inverse or limiting points with respect to the circle. Obviously any circle has an infinite number of pairs of inverse points in a given straight line through the center. Two given circles will have one pair of inverse points in common.

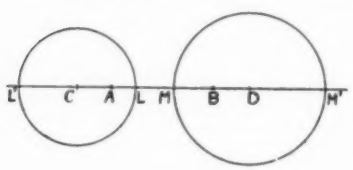

Fig. 1.

Let A and B be the inverse points, C and D the centers of the circles, L, L', M, M' the points where the line of centers cuts the circles (Fig. 1).

Let $CL = a$, $MD = b$.

Let $AB = 2c$, then $CB \cdot CA = a^2$, or $CB(CB - 2C) = a^2$.

Solving for CB,

$$CB = C + \sqrt{a^2 + c^2}. \tag{1}$$

Also,

$$DB(DB + 2c) = b^2.$$

Solving for DB,

$$DB = -c + \sqrt{b^2 + c^2}. \tag{2}$$

If $d = CB$, we get by adding (1) and (2)

$$d = \sqrt{a^2 + c^2} + \sqrt{b^2 + c^2}. \tag{3}$$

Solving (3) for c,

$$c^2 = \frac{4}{d^2}(d + a + b)(d + a - b)(d + b - a)(d - a - b). \tag{4}$$

Equations (1), (2) and (4) locate the point B.

To find position of A we have

$$CA = CB - 2c = \sqrt{a^2 + c^2} - c, \tag{5}$$

$$DA = d - AC = \sqrt{b^2 + c^2} + c. \tag{6}$$

When $a = b$, $d = 2\sqrt{a^2 + c^2}$, and

$$c = \sqrt{\frac{d^2}{4} - a^2}. \tag{7}$$

The Equations of the First Transformation of Laplace's Equation.—The potential at any point satisfies Laplace's equation which in cylindrical coördinates is

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{z}\frac{\partial V}{\partial z} = 0, \tag{8}$$

where z is the distance of the point from the axis or line of centers, and r is the distance of the point from any fixed plane normal to the axis.

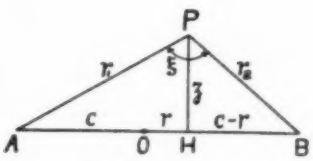

Fig. 2.

Let r_1, r_2 denote the distances of the point from the inverse points of the sheres, e^η the ratio of these distances, ξ the angle between them. Then (Fig. 2)

$$e^\eta = \frac{r_2}{r_1} \quad \text{or} \quad \eta = \log_e\left(\frac{r_2}{r_1}\right) \tag{9}$$

and

$$4c^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\xi$$

or

$$\cos\xi = \frac{r_1^2 + r_2^2 - 4c^2}{2r_1r_2}. \tag{10}$$

Taking origin at middle point of AB,

$$r_1^2 = z^2 + (c + r)^2, \quad r_2^2 = z^2 + (c - r)^2. \tag{11}$$

From (11)

$$r_1^2 + r_2^2 - 4c^2 = 2(z^2 + r^2 - c^2), \tag{12}$$

$$r_1^2r_2^2 = (z^2 + r^2 + c^2)^2 - 4r^2c^2$$

$$= (z^2 + r^2 - c^2)^2 + 4c^2z^2. \tag{13}$$

Then by (10),

$$\cos \xi = \frac{z^2 + r^2 - c^2}{\sqrt{(z^2 + r^2 - c^2)^2 + 4c^2z^2}}$$

and

$$\tan \xi = \frac{2cz}{z^2 + r^2 - c^2}. \tag{14}$$

From (9)

$$\eta = \tfrac{1}{2} \log \left[\frac{z^2 + (c + r)^2}{z^2 + (c - r)^2} \right]. \tag{15}$$

From (14) and (15)

$$\frac{r^2 + z^2 - c^2}{2cz} = \cot \xi, \quad \text{and} \quad \frac{r^2 + z^2 + c^2}{2cr} = \coth \eta. \tag{16}$$

From (16)

$$2cz \cot \xi = r^2 + z^2 - c^2, \quad \text{and} \quad 2cr \coth \eta = r^2 + c^2 + z^2. \tag{17}$$

Subtracting and removing factor $2c$,

$$r(\coth \eta - \cot \xi) = c. \tag{18}$$

From (17)

$$r^2 \operatorname{cosech}^2 \eta = \frac{(r^2 + c^2 + z^2)^2 - 4c^2r^2}{4c^2}, \quad z^2 \operatorname{cosec}^2 \xi = \frac{(r^2 + z^2 + c^2) - 4c^2r^2}{4c^2}.$$

Therefore

$$r \operatorname{cosech} \eta = z \operatorname{cosec} \xi. \tag{19}$$

Solving (18) and (19),

$$r = \frac{c \sinh \eta}{\cosh \eta - \cos \xi}, \quad \text{and} \quad z = \frac{c \sin \xi}{\cosh \eta - \cos \xi}. \tag{20}$$

Partial Derivatives of η and ξ with Respect to r and z.—The functions η and ξ are conjugate functions as will be seen when the derivatives are written.

$$\frac{\partial \eta}{\partial r} = \frac{2c\,(z^2 - r^2 + c^2)}{(z^2 + r^2 + c^2)^2 - 4r^2c^2}, \quad \frac{\partial \xi}{\partial z} = \frac{-\,2c(z^2 - r^2 + c^2)}{(z^2 + r^2 + c^2)^2 - 4r^2c^2}, \tag{21}$$

$$\frac{\partial \eta}{\partial z} = \frac{-\,4czr}{(r^2 + z^2 + c^2)^2 - 4r^2c^2}, \quad \frac{\partial \xi}{\partial r} = \frac{-\,4czr}{(r^2 + z^2 + c^2)^2 - 4r^2c^2}. \tag{22}$$

Equations (21) and (22) show that

$$\frac{\partial \eta}{\partial r} = -\frac{\partial \xi}{\partial z} \quad \text{and} \quad \frac{\partial \eta}{\partial z} = \frac{\partial \xi}{\partial r}. \tag{23}$$

From (23) we easily obtain

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2} = 0, \quad \text{and} \quad \frac{\partial^2 \xi}{\partial r^2} + \frac{\partial^2 \xi}{\partial z^2} = 0. \tag{24}$$

Also

$$\left(\frac{\partial \eta}{\partial r}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2 = \left(\frac{\partial \xi}{\partial r}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2 = \frac{4c^2}{(r^2 + z^2 + c^2)^2 - 4r^2c^2}. \quad (25)$$

First Transformation of Laplace's Equation.—The relations (24) and (25) make the transformation quite easy.

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial V}{\partial \xi} \cdot \frac{\partial \xi}{\partial z}, \quad (26)$$

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial \eta} \cdot \frac{\partial \eta}{\partial r} + \frac{\partial V}{\partial \xi} \cdot \frac{\partial \xi}{\partial r}, \quad (27)$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial z}\right)^2 + \frac{\partial V}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 V}{\partial z^2} \cdot \left(\frac{\partial \xi}{\partial z}\right)^2 + \frac{\partial V}{\partial \xi} \cdot \frac{\partial^2 \xi}{\partial z^2}, \quad (28)$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial r}\right)^2 + \frac{\partial V}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 V}{\partial \xi^2} \cdot \left(\frac{\partial \xi}{\partial r}\right)^2 + \frac{\partial V}{\partial \xi} \cdot \frac{\partial^2 \xi}{\partial r^2}. \quad (29)$$

Adding (28) and (29) and using (24) and (25)

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} = \frac{4c^2}{(r^2 + z^2 + c^2)^2 - 4c^2r^2} \cdot \left\{ \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} \right\}. \quad (30)$$

From (26) and (21) and (22)

$$\frac{1}{z}\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \eta}\left(\frac{-4cr}{(r^2 + z^2 + c^2) - 4c^2r^2}\right) + \frac{\partial V}{\partial \xi}\left(\frac{-2c(r^2 - z^2 - c^2)}{z[(r^2 + z^2 + c^2)^2 - 4c^2r^2]}\right). \quad (31)$$

Adding (30) and (31) and removing factor $4c^2/(r^2 + z^2 + c^2)^2 - 4c^2r^2$,

$$\frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{r}{c} \cdot \frac{\partial V}{\partial \eta} + \frac{r^2 - z^2 - c^2}{2cz} \cdot \frac{\partial V}{\partial \xi} = 0. \quad (32)$$

It remains to substitute the value of r/c and of $r^2 - z^2 - c^2/2cz$ in terms of η and ξ. The value of r and of z are given by (20).

$$\frac{r}{c} = \frac{\sinh \eta}{\cosh \eta - \cos \xi}, \qquad \frac{r^2 - z^2 - c^2}{2cz} = \frac{\cosh \eta \cos \xi - 1}{\sin \xi (\cosh \eta - \cos \xi)}.$$

Then (32) becomes

$$\frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \xi^2} - \frac{\sinh \eta \sin \xi}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial V}{\partial \eta} + \frac{\cosh \eta \cos \xi - 1}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial V}{\partial \xi} = 0. \quad (33)$$

Second Transformation.—Let $V = U\sqrt{2(\cosh \eta - \cos \xi)}$. For convenience of manipulation put $\sqrt{2(\cosh \eta - \cos \xi)} = F$.

$$\frac{\partial F}{\partial \eta} = \frac{\sinh \eta}{F}, \quad \frac{\partial^2 F}{\partial \eta^2} = \frac{F^2 \cosh \eta - \sinh^2 \eta}{F^3}, \tag{34}$$

$$\frac{\partial F}{\partial \xi} = \frac{\sin \xi}{F}, \quad \frac{d^2 F}{\partial \xi^2} = \frac{F^2 \cos \xi - \sin^2 \xi}{F^3}, \tag{35}$$

$$\frac{\partial V}{\partial \eta} = F\frac{\partial U}{\partial \eta} + U\frac{\partial F}{\partial \eta} = F\frac{\partial U}{\partial \eta} + \frac{U \sinh \eta}{F}. \tag{36}$$

$$\frac{\partial V}{\partial \xi} = F\frac{\partial U}{\partial \xi} + U\frac{\partial F}{\partial \xi} = F\frac{\partial U}{\partial \xi} + \frac{U \sin \xi}{F}, \tag{37}$$

$$\frac{\partial^2 V}{\partial \eta^2} = F\frac{\partial^2 U}{\partial \eta^2} + \frac{2 \sinh \eta}{F}\frac{\partial U}{\partial \eta} + U\left\{\frac{F^2 \cosh \eta - \sinh^2 \eta}{F^3}\right\}, \tag{38}$$

$$\frac{\partial^2 V}{\partial \xi^2} = F\frac{\partial^2 U}{\partial \xi^2} + \frac{2 \sin \xi}{F}\frac{\partial U}{\partial \xi} + U\left\{\frac{F^2 \cos \xi - \sin^2 \xi}{F^3}\right\}. \tag{39}$$

Substituting (36), (37), (38), (39) in (33) and collecting coefficients

$$F\left(\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2}\right) + \frac{2 \sinh \eta - 2 \sinh \eta}{F}\frac{\partial U}{\partial \eta} + \frac{2(\sin^2 \xi + \cosh \eta \cos \xi - 1)}{F \sin \xi}\frac{\partial U}{\partial \xi}$$
$$+ U\left\{\frac{F^2(\cosh \eta + \cos \xi) - 2 \sinh^2 \eta - \sin^2 \xi + 2(\cosh \eta - \cos \xi) - 2}{F^3}\right\} = 0,$$

which reduces to

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \xi^2} + \cot \xi \frac{\partial U}{\partial \xi} - \frac{U}{4} = 0. \tag{40}$$

Third Transformation.—If we put $U = RQ$, where R is a function of η alone and Q a function of ξ alone, we get

$$Q\frac{d^2 R}{d\eta^2} + R\frac{d^2 Q}{\partial \xi^2} + \cot \xi R\frac{dQ}{d\xi} - \frac{RQ}{4} = 0.$$

Dividing by RQ and separating functions of η from those of ξ.

$$\left(\frac{1}{R}\frac{d^2 R}{d\eta^2} - \frac{1}{4}\right) = -\frac{1}{Q}\left(\frac{d^2 Q}{d\xi^2} + \cot \xi \frac{dQ}{d\xi}\right). \tag{41}$$

The left member of (41) does not contain ξ, and the right member does not contain η, hence they must each be equal to a constant which it is convenient to denote by $m(m+1)$, m being any positive integer.

Then

$$\frac{d^2R}{d\eta^2} - (m^2 + m + \tfrac{1}{4})R = 0, \quad \text{or} \quad \frac{d^2R}{\partial\eta^2} - (m + \tfrac{1}{2})^2R = 0. \qquad (42)$$

Particular solutions of this equaltion are $e^{(m+1/2)\eta}$ and $e^{-(m+1/2)\eta}$. Also

$$\frac{d^2Q}{d\xi^2} + \cot\xi\frac{dQ}{d\xi} + m(m+1)Q = 0. \qquad (43)$$

Equation (43) is satisfied by $P_m(\cos\xi)$ when m is a positive integer, $P_m(\cos\xi)$ denoting a zonal harmonic or Legendre function. (See Byerly's Fourier Series and Spherical Harmonics, Chapter I.) Then

$$U = \sum_0^\infty A_m R_{1m} P_m(\cos\xi) + \sum_0^\infty B_m R_{2m} P_m(\cos\xi) \qquad (44)$$

and

$$V = U\sqrt{2(\cosh\eta - \cos\xi)}, \qquad (45)$$

where R_{1m} and R_{2m} are particular solutions of (43) which fulfill certain requirements, and A_m and B_m are constants in certain special expansions. The requirements are the following:

U shall reduce to $\dfrac{V_1}{\sqrt{2(\cosh\alpha - \cos\xi)}}$ when $\eta = \alpha$ for all values of ξ;

U shall reduce to $\dfrac{V_2}{\sqrt{2(\cosh\beta - \cos\beta)}}$ when $\eta = -\beta$ for all values of ξ;

R_{1m} is to be determined so that its value is zero when $\eta = -\beta$ and unity when $\eta = \alpha$;

R_{2m} is to determined so that its value is zero when $\eta = \alpha$ and unity when $\eta = -\beta$.

The particular solutions $e^{(m+1/2)\eta}$ and $e^{-(m+1/2)\eta}$ of the equation

$$\frac{d^2P}{d\eta^2} - (m + \tfrac{1}{2})^2P = 0$$

may be multiplied by any constants, and added or subtracted and the results will still be solutions. Then $e^{-(m+1/2)\alpha} \cdot e^{+(m+1/2)\eta}$ and $e^{(m+1/2)\alpha} \times e^{-(m+1/2)\eta}$ are solutions. These are the same as $e^{-(m+1/2)(\alpha-\eta)}$ and $e^{(m+1/2)(\alpha-\eta)}$. Then $\frac{1}{2}\{e^{+(m+1/2)(\alpha-\eta)} - e^{-(m+1/2)(\alpha-\eta)}\}$ is a solution which vanishes for $\eta = \alpha$. It is identical with $\sinh(m + \frac{1}{2})(\alpha - \eta)$. The function

$$\frac{\sinh(m + \frac{1}{2})(\alpha - \eta)}{\sinh(m + \frac{1}{2})(\alpha + \beta)}$$

is equal to unity when $\eta = -\beta$. In like fashion

$$\frac{\sinh (m + \frac{1}{2})(\beta + \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)}$$

vanishes when $\eta = -\beta$ and is equal to unity when $\eta = \alpha$. We have then

$$R_{1m} = \frac{\sinh (m + \frac{1}{2})(\beta + \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)} \quad \text{and} \quad R_{2m} = \frac{\sinh (m + \frac{1}{2})(\alpha - \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)}.$$

When $\eta = \alpha$, we have

$$\frac{V_1}{\sqrt{2}(\cosh \alpha - \cos \xi)} = \sum_0^\infty A_m P_m (\cos \xi),$$

because R_{1m} is unity.

To find the values of A_m, the function $\{2(\cosh \alpha - \cos \xi)\}^{-1/2}$ must be expanded in a series of zonal harmonics.

$P_m(x)$ is defined as the coefficient of z^m in the expansion of $(1 - 2xz + z^2)^{-1/2}$. The function $\{2(\cosh \alpha - \cos \xi)\}^{-1/2}$ is easily thrown into this form, for we have $2 \cosh \alpha = e^\alpha + e^{-\alpha}$, and

$$\{e^\alpha + e^{-\alpha} - 2 \cos \xi\}^{-1/2} = e^{-\alpha/2}\{1 - 2e^{-\alpha} \cos \xi + e^{-2\alpha}\}^{-1/2}.$$

Then $P_m(\cos \xi)$ is the coefficient of $e^{-m\alpha}$ in the expansion of

$$\{1 - 2e^{-\alpha} \cos \xi + e^{-2\alpha}\}^{-1/2}.$$

Consequently

$$\{2 (\cosh \alpha - \cos \xi)\}^{-1/2} = e^{-\alpha/2} \sum_0^\infty e^{-m\alpha} P_m (\cos \xi),$$

$$= \sum_0^\infty e^{-(m+1/2)\alpha} P_m (\cos \xi).$$

Obviously,

$$A_m = V_1 e^{-(m+1/2)\alpha}$$

and

$$B_m = V_2 e^{-(m+1/2)\beta}.$$

Substituting the values of A_m, B_m, R_{1m} and R_{2m} in the formula

$$V = \sum_0^\infty A_m R_{1m} P_m (\cos \xi) + \sum_0^\infty B_m R_{2m} P_m (\cos \xi)$$

we get the formula at the beginning of the paper.

FORMULÆ FOR SPECIAL VALUES OF V_1 AND V_2.

Equal Spheres, Potentials $V_1/2$ and $-V_1/2$.—Here $\alpha = \beta$, then

$$V = \frac{V_1}{2}\sqrt{2(\cosh\eta - \cos\xi)}\sum_0^\infty \frac{\sinh(m+\frac{1}{2})(\alpha+\eta) - \sinh(m+\frac{1}{2})(\alpha-\eta)}{\sinh(m+\frac{1}{2})(\alpha+\alpha)} \cdot P_m(\cos\xi)e^{-(m+1/2)\alpha}.$$

Since $\sinh A - \sinh B = 2\sinh\frac{1}{2}(A-V)\cos\frac{1}{2}(A+B)$, the numerator becomes $2\sinh\left(\frac{2m+1}{2}\right)\eta\cosh(m+\frac{1}{2})\alpha$, and then

$$V = \frac{V_1}{2}\sqrt{2(\cosh\eta - \cos\xi)}\sum_0^\infty \frac{\sinh(m+\frac{1}{2})\eta}{\sinh(m+\frac{1}{2})\alpha} \cdot P_m(\cos\xi)e^{-(m+1/2)\alpha}. \quad (46)$$

It is easily seen that this reduces to $V_1/2$, when $\eta = \alpha$.

Equal Spheres, Potentials V_1 and 0.—Again $\alpha = \beta$, and

$$V = V_1\{2(\cosh\eta - \cos\xi)\}^{1/2}\sum_0^\infty \frac{\sinh(m+\frac{1}{2})(\alpha+\eta)}{\sinh(m+\frac{1}{2})(2\alpha)} P_m(\cos\xi)e^{-(m+1/2)\alpha}. \quad (47)$$

These appear to be the only cases of interest to physicists.

II. THE ELECTROSTATIC FORCE OR VOLTAGE GRADIENT.

At any point in the line of centers, between the spheres, $\cos\xi = -1$. Then, for the voltage gradient G, we have

$$G = \frac{\partial V}{\partial\eta}\cdot\frac{\partial\eta}{\partial r}. \quad (48)$$

In this special case $z = 0$, and

$$\eta = \log\frac{c+r}{c-r},$$

by (3). Then

$$\frac{\partial\eta}{\partial r} = \frac{2c}{c^2 - r^2}. \quad (49)$$

Let x = distance between surfaces, then $x + 2a = \sqrt{a^2 + c^2}$, by (7). Then

$$2c = \sqrt{x^2 + 4ax},$$

and

$$\frac{\partial\eta}{\partial r} = \frac{\sqrt{x^2 + 4ax}}{ax},$$

since $r = x/2$ at surface. Also

$$e^{\alpha} = \frac{\sqrt{x^2 + 4ax} + x}{\sqrt{x^2 + 4ax} - x},$$

whence

$$\frac{\sqrt{x^2 + 4ax}}{ax} = \frac{1}{a} \coth \frac{\alpha}{2}.$$

In the case of equal spheres, potentials $+ V_1/2$ and $- V_1/2$, we have

$$V = \frac{V_1}{2} \sqrt{2(\cosh \eta - \cos \xi)} \sum_0^\infty \frac{\sinh (m + \frac{1}{2})\eta}{\sinh (m + \frac{1}{2})\alpha} P_m (\cos \xi) e^{-(m+1/2)\alpha}.$$

$$\frac{\partial V}{\partial \eta} = \frac{V_1}{2} \left\{ \sqrt{2(\cosh \eta - \cos \xi)} \sum_0^\infty \frac{\cosh (m+\frac{1}{2})\eta}{\sinh (m+\frac{1}{2})} \cdot (m+\tfrac{1}{2}) P_m (\cos \xi) e^{-(m+1/2)\alpha} \right.$$

$$\left. + \frac{\sinh \eta}{\sqrt{2(\cosh \eta - \cos \xi)}} \sum_0^\infty \frac{\sinh (m + \frac{1}{2})\eta}{\sinh (m + \frac{1}{2})\alpha} P_m (\cos \xi) e^{-(m+1/2)\alpha} \right\}.$$

When $\eta = \alpha$, and $\cos \xi = -1$,

$$\left(\frac{\partial V}{\partial \eta}\right)_{\eta=\alpha} = \frac{V_1}{2} \left\{ \sinh \frac{\alpha}{2} \sum_0^\infty P_m(-1) e^{-(m+1/2)\alpha} \right.$$

$$\left. + 2 \cosh \frac{\alpha}{2} \sum_0^\infty (m + \tfrac{1}{2}) \coth (m + \tfrac{1}{2})\alpha P_m(-1) e^{-(m+1/2)\alpha} \right\}.$$

$$\sum_0^\infty P_{m(-1)} e^{-(m+1/2)\alpha} = e^{-\alpha/2}(1 - e^{-\alpha} + e^{-2\alpha} - e^{-3\alpha} + \cdots)$$

$$= e^{-\alpha/2}(1 + e^{-\alpha})^{-1} = \frac{1}{2 \cosh \frac{\alpha}{2}}.$$

Then

$$\left(\frac{\partial V}{\partial \eta}\right)_{\eta=\alpha} = V \left\{ \frac{1}{4} \tanh \frac{\alpha}{2} + \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \frac{\alpha}{2} - \frac{3}{2} e^{-3\alpha/} \coth \frac{3\alpha}{2} \right.\right.$$

$$\left.\left. + \frac{5}{2} e^{-5\alpha/2} \coth \frac{5\alpha}{2} - \frac{7}{2} e^{-7\alpha/2} \coth \frac{7\alpha}{2} + \cdots \right] \right\}$$

and

$$G = \frac{V_1}{a} \left\{ \frac{1}{4} + \coth \frac{\alpha}{2} \cosh \frac{\alpha}{2} \left[\frac{1}{2} e^{-\alpha/2} \coth \frac{\alpha}{2} - \frac{3}{2} e^{-3\alpha/2} \coth \frac{3\alpha}{2} + \cdots \right] \right\}. \quad (50)$$

The series $\sum_0^\infty (m + \frac{1}{2}) e^{-(m+1/2)\alpha} \coth (m + \frac{1}{2})\alpha$ is highly convergent for large values of α, but very slowly convergent for small values of α.

In practice the values of x/a are given, the values of α calculated, and the values of the functions taken out of tables of hyperbolic and exponential functions.

We have

$$\alpha = \log_e \frac{\sqrt{x^2 + 4ax} + x}{\sqrt{x^2 + 4ax} - x}.$$

For example if the value of x/a is $\frac{1}{2}$, we get $\alpha = \log_e 2 = 2.3025 \log_{10} 2 = 0.69314$. When x/a is very large, α is very large, and then we have, very nearly, $\coth \alpha/2 = 1$, $\coth 3\alpha/2 = 1$, etc., and $\cosh \alpha/2 = \frac{1}{2}e^{\alpha/2}$, and G reduces to

$$G = \frac{V_1}{a}\{\tfrac{1}{4} + \tfrac{1}{4}[1 - 3e^{-\alpha} + 5e^{-2\alpha} - 7e^{-3\alpha} + \cdots]\}$$
$$= \frac{V_1}{a}\left\{\frac{1}{4} + \frac{1}{4}\left[\frac{1 - e^{-\alpha}}{(1 + e^{-\alpha})^2}\right]\right\} = \frac{V}{x}\left\{\frac{1}{2}\left(\frac{x}{a} + 1\right)\right\}. \qquad (51)$$

When the equal spheres have potentials V_1 and 0, the potential is

$$V = V_1\sqrt{2(\cosh \eta - \cos \xi)}\sum_0^\infty \frac{\sinh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})(\alpha + \beta)} P_m(\cos \xi)e^{-(m+1/2)\alpha}.$$

Then

$$\frac{\partial V}{\partial \eta} = V_1\left\{\frac{\sinh \eta}{\sqrt{2(\cosh \eta - \cos \xi)}}\sum_0^\infty \frac{\sinh (m + \frac{1}{2})(\alpha + \eta)}{\sinh (m + \frac{1}{2})(2\alpha)} P_m'(\cos \xi)e^{-(m+1/2)\alpha}\right.$$
$$\left. + \sqrt{2(\cosh \eta - \cos \xi)}\sum_0^\infty \frac{(m+\frac{1}{2})\cosh (m+\frac{1}{2})(\alpha+\eta)}{\sinh (m + \frac{1}{2})2\alpha} P_m(\cos \xi)^{-(m+1/2)\alpha}\right\}.$$

When $\eta = \alpha$, and $\cos \xi = -1$,

$$\left(\frac{\partial V}{\partial \eta}\right)_{\eta=\alpha} = V_1\left\{\sinh \frac{\alpha}{2}\sum_0^\infty P_m(-1)e^{-(m+1/2)\alpha}\right.$$
$$\left. + 2\cosh \frac{\alpha}{2}\sum_0^\infty (m + \tfrac{1}{2})P_m(-1)e^{-(m+1/2)\alpha}\coth (2m + 1)\alpha\right\}$$
$$= \frac{V}{2}\left\{\tanh \frac{\alpha}{2} + 4\cosh \frac{\alpha}{2}[\tfrac{1}{2}e^{-\alpha/2}\coth \alpha - \tfrac{3}{2}e^{-3\alpha/2}\coth 3\alpha + \cdots]\right\},$$

and

$$G = \frac{V}{a}\left\{\frac{1}{2} + 2\coth \frac{\alpha}{2}\cosh \frac{\alpha}{2}[\tfrac{1}{2}e^{-\alpha/2}\coth \alpha - \tfrac{3}{2}e^{-3\alpha/2}\coth 3\alpha + \cdots]\right\}.$$

This series like the other is rapidly convergent for large values of α, but very slowly for small values. The limiting value of G for large values is $\frac{V}{x}\left\{\frac{x}{a} + \frac{a}{x}\right\}$.

AN ATTEMPT TO DETECT POSSIBLE CHANGES IN WEIGHT OR MOMENTUM EFFECTS ON CHARGING A CONDENSER.

By P. G. Agnew and W. C. Bishop.

THERE is no reason to suppose that the weight of a body is affected by its state of electrification in great enough a degree to be detected by the methods of direct weighing at present available. Yet however improbable such an effect may seem, the question cannot be settled without an appeal to experiment. Similar investigations have been made to detect possible changes in weight due to temperature[1] and to magnetization,[2] which have, as was anticipated, given negative results.

Southerns has attacked the problem by inducing opposite charges at the ends of the beam of a balance (which was earthed) by means of an external electrostatic field and noting the difference in readings on reversing the external field.[3]

A small but definite residual appeared in his results which he was unable to account for by experimental errors. He makes it clear, however, that he does not regard it as an established experimental fact that the weight of a body depends upon its electrification, as the sensitivity had been pushed to an extreme, and consequently the small residual may have been due to a systematic error. In fact, unless the result is confirmed by further work this must remain the most probable explanation. He says, "Whether the effect be due to a modification of gravity or not, the experiments appear to show that a body, positively charged with 20 electrostatic units of electricity, behaves as though it were heavier than the same body negatively charged, by an amount of the order of .0007 mgm."

Change of Weight.

It seemed worth while to make some weighings on a condenser with a good balance to see whether any difference in weight could be detected between the charged and the discharged conditions. The experiment is analogous but not equivalent to Southerns's since both plates of the condenser are in this case necessarily on the same pan of the balance and hence if the positive plate loses in weight the same amount that the

[1] Southerns, Proc. Roy. Soc., A *78*, p. 392, 1906.

[2] Lloyd, Terrestrial Magnetism, *14*, p. 67, 1909.

[3] Southerns, Proc. Camb. Phil. Soc., *15*, p. 352, 1909.

negative plate gains, or vice versa, it could not be detected. There is the advantage, however, that very much greater charges may be used in the light paper condensers now available. And further, the external electrostatic field may be made extremely small, thus minimizing the chance of extraneous effects entering.

Since all the results are negative, the work will be described but briefly.

After preliminary work with other types, the final weighings were made with five 2 mf. condensers of the Mansbridge type. They were joined together and shielded as a single condenser, a hole just sufficient for the leading in wire being left in the tin foil shield to which the one plate was connected. In the final weighings, this plate and the metallic parts of the balance, including the levelling screws, were earthed irrespective of the condition of charge of the condenser. In order to be able to charge and discharge the condenser without arresting the balance, a flexible connection was made by two 0.02 mm. silver wires about 6 cm. long. These gave no trouble whatever in decreasing the sensitivity of the balance, which was a Becker type designed for a load of a kilogram. To avoid possible heating of the leading-in wires the condenser was always charged and discharged through a resistance of 500 ohms. Electromagnetic control of the amplitude of the swing was found very convenient. For this purpose iron was used as a part of the counterweight and a small electromagnet placed under the balance case. The amplitude could easily be controlled to a few tenths of a mm.

In taking the observations 5 turning points were read for each rest point. The readings were taken as follows. With the positive pole of the battery connected to earth 5 rest points were taken, the condenser being charged and discharged alternately between rest points, and the whole then repeated with the negative pole of the battery to earth. Such a set of observations will be referred to as a group.

After a considerable number of preliminary difficulties were overcome no group of readings was made under satisfactory conditions which showed a greater difference than 0.1 mm. on the scale between the average of the 6 rest points taken with the condenser charged and the 4 rest points taken with the condenser uncharged. This corresponds to 0.015 mg.

The average with the + pole of the battery to earth was never quite the same as that with the − pole to earth. This difference, which was nearly always less than 0.1 mm. on the scale, differed in direction from evening to evening, but was usually in the same direction throughout a given evening. It was probably due to small charges within the balance case acting on the leading-in wires. It was decreased by placing radio-

active matter in the case, and by leaving the balance undisturbed for some time before taking readings, but it never entirely disappeared.

There seemed to be no particular advantage in taking any very large number of observations, so on an evening when conditions were the very best four groups of readings were taken in a continuous series during which time neither observer so much as moved from his position in order that no disturbing factor might enter. No reading was lost and none have been omitted in calculating the results given below. The final average depends upon 4 groups during which 40 rest points or 200 turning points were taken.

Capacity of condenser	10 mf.
Voltage	240
Quantity (coulombs)	$2.4 \cdot 10^{-3}$
Quantity (electrostatic units)	$7.2 \cdot 10^{8}$
Weight of condensers	930 g.
Scale distance	3.1 meters
Period of balance	45 sec.
Sensitivity, 1 cm. of scale =	1.5 mg.
Av. rest point, condensers charged	84.253
Av. rest point, condensers discharged	84.255
Difference =	0.002 cm.
Apparent increase in weight =	0.003 mg.

It is seen that the actual average difference in weight found is one part in 300,000,000, but it is probably better to estimate the accuracy as somewhat lower, say in round numbers, 0.01 mg. or one part in 100,000,000

As already pointed out our negative result does not necessarily conflict with the apparent small positive effect found by Southerns, since his experiment attempted to distinguish between the weight of a body when positively and when negatively charged. If the positive plate increases and the negative plate decreases by an exactly equal amount Southerns's experiment would give a positive result while a condenser would not change its weight upon charge or discharge. But if the plates change in weight by amounts which are not equal and opposite, the weight of the whole condenser must, of course, be changed. If we assume the small difference found by Southerns to be a true effect and to be proportional to the charge, the difference in weight per electrostatic unit would be $3.5 \cdot 10^{-8}$ g., or 100 g. per coulomb. According to the present experiment the change per electrostatic unit is less than $1.4 \cdot 10^{-12}$ g. or less than 4 mg. per coulomb.

The electromagnetic mass is of course a quantity of an entirely different order of magnitude than that which may be reached by the modern precision balance; w/c^2 in the present case is but the billionth part of the actual residual found (0.003 mg.).

MOMENTUM.

Inasmuch as the same arrangement of apparatus sufficed for the purpose, an attempt was made to detect possible momentum effects upon charge and discharge of the condenser. A condenser standing upright on the scale pan and charged from above might receive a kick on charge and an opposite kick on discharge and, however improbable, it was decided also to look for the possibility of a kick which would be in the same direction on both charge and discharge.

The method used was to time the charge and discharge so that any momentum effect would tend to act cumulatively in changing the amplitude of swing. In order to look for the first kind of an effect, the balance was given a small amplitude of swing and the condenser charged during 10 swings to the right, and discharged during the 10 swings to the left, and then the process reversed for 10 complete swings. It is evident thaf if there is such an effect as looked for the first half of the process ought to tend to either increase or to decrease the amplitude, while the opposite effect should take place during the second half.

A group of actual readings will make the matter clear and at the same time show how the effect of damping can be eliminated.

March 15, 1912—to Earth.

Turning Point.	Double Amplitude.	
8.91		
8.11	.80	Charged on increasing and discharged on decreasing swings.
8.90	.79	
8.12	.78	
8.89	.77	
8.13	.76	
8.88	.75	
8.18	.70	
8.87	.69	
8.20	.67	
8.88	.68	
8.22		
8.88	.66	Charged on decreasing and discharged on increasing swings.
8.23	.65	
8.87	.64	
8.24	.63	
8.83	.59	
8.24	.59	
8.81	.57	
8.25	.56	
8.80	.55	
8.28	.52	

It was found that for the small amplitudes the damping was small enough so that a sensibly straight line was obtained by plotting amplitude against time. Hence if there is no true momentum effect we should get a constant by adding the first 10 amplitudes to the second 10 amplitudes taken in reversed order, that is, adding .80 to .52, .79 to .55 and so on,

but if there is a momentum effect then these numbers should either constantly increase or constantly decrease. Such an addition gives

1.32
1.34
1.34
1.34
1.35
1.34
1.33
1.33
1.32
1.34

It is seen that these numbers are constant within the errors of observation. In a careful examination of all the runs taken under favorable conditions (after various disturbing factors had been removed), no tendency either to increase or to decrease the amplitude could be detected. A conservative estimate of the amount of change in amplitude which could have been readily and unmistakably detected is 0.04 cm. for the group of 10 swings or 0.004 cm. per swing.

In order to interpret this change in amplitude in C.G.S. units of momentum, we must consider the constants of the balance when swinging through a small angle. It may easily be shown that a given change in amplitude means a given change in momentum, no matter what the amplitude is, the maximum angular momentum being proportional to the square of the amplitude, just as the maximum kinetic energy is proportional to the square of the amplitude.

Let A = amplitude of swing (radians).
A_c = amplitude of swing (cm.).
f = force in dynes necessary to give 1 cm. deflection when applied to a pan.
I = equivalent moment of inertia of moving parts.
K = restoring torque.
l = length of balance arm.
M = momentum.
s = scale distance.
T = complete period of balance.
U = angular momentum at middle of swing.
θ = angle of displacement.

Then

$$T = 2\pi\sqrt{\frac{I}{K}}, \tag{1}$$

$$\theta = A\cos\frac{2\pi}{T}t$$

and

$$U = I\dot{\theta}_{\max} = \frac{2\pi IA}{T}. \tag{2}$$

Eliminating I from (1) and (2)

$$U = \frac{KT}{2\pi} A,$$

which shows that the maximum angular momentum is proportional to the amplitude. Now if a small momentum ΔM is added to one pan the angular momentum is changed by

$$\Delta U = l\Delta M,$$

$$\therefore \Delta M = \frac{KT}{2\pi l} \Delta A. \tag{3}$$

Since deflections are read by a mirror 1 cm. is equivalent to $1/2s$ radians. Hence the torque per radian is

$$K = 2sfl.$$

We also have

$$A = \frac{A_e}{2s},$$

$$\therefore \Delta M = \frac{fT}{2\pi} \Delta A_e.$$

In the present case $f = 1.5$, $T = 45$, $\Delta A_e = 0.001$ (since a difference of 0.004 cm. could have been detected in the sum of two double amplitudes). This gives for the momentum that could have been detected

$$\Delta M = 0.01 \text{ dyne-sec. (approximately).}^1$$

We may then say that the experiment shows that in charging a 10 mf. condenser to 240 volts, any momentum effect of the first kind postulated is less than 0.01 dyne-second.

In looking for the second kind of momentum effect, that is, an impulse in the same direction on both charge and discharge, the same general method was used. In this case, however, the condenser could be charged and discharged on alternate swings, or the sensitivity could be increased by charging and discharging several times each swing. It was found that 15 charges and discharges, or 30 operations per swing, could be easily

[1] Barrell has recently suggested the use of "second-dyne" as a name for the C.G.S. unit of momentum (Nature, *87*, p. 144, 1911). An inversion to "dyne-second" seems decidedly better as it is more euphonious and more easily spoken.

carried out without trouble arising from heat being developed in the leading-in wires.[1]

In this case also the result was negative although as before a change of 0.004 cm. in double amplitude per swing could certainly have been detected. Since 15 charges and discharges per swing were used, this means that the momentum of a charge or discharge that could have been detected was 1/30 of that in the first case, or approximately 0.0004 dyne-sec.

The results of the tests for momentum effects were independent of the position of condenser on the pan, whether upright, inverted or lying on its side.

SUMMARY.

1. Charging a 10 mf. condenser to 240 volts was found not to change its weight by as much as 0.01 mg., or one part in 100,000,000.

2. In charging and discharging the same condenser no momentum effect having opposite signs on charge and on discharge as great as 0.01 dyne-sec. was detected.

3. Similarly no momentum effect having the same sign on charge and on discharge as great as 0.0004 dyne-sec. was detected.

BUREAU OF STANDARDS,
WASHINGTON, D. C.

[1] It may be interesting to note that a contact resistance at the condenser gave a very decided simulation of the momentum effect of the second kind by the air currents set up. Hence soldered connections were used.

A GENERAL PROBLEM IN GYROSTATIC ACTION.

By J. W. Milnor.

THE theory of gyrostatic action has been prominent in recent technical writings due to the development of the monorail car. In the January number of the Physical Review Professor Franklin discussed the behavior of an unconstrained gyrostat under a suddenly applied torque. The motion due to a torque proportional to the time was calculated by B. L. Newkirk in the July number. In the following paper an approximate solution in series is given for the motion caused by any applied torque whatever, where this torque is known as a function of the time.

In the practical form in which we are interested, the gyrostat consists of a wheel symmetrical about its axis, and rotating at a high rate of speed about that axis. The wheel is so mounted that its axis is free to turn in any direction. If the axis of such an arrangement be constrained to turn in a given direction, it will exert a force on the constraining mechanism at right angles to the plane in which it turns. Conversely, if an external torque acts so as to tend to change the direction of the axis of a free gyrostat, the gyrostat will resist the torque. At the same time the axis of the gyrostat will start to turn slowly about an axis at right-angles to the axis about which the disturbing torque acts, and the torque due to this precession of the axis will balance the external torque. While the precession is starting the gyrostat will yield slightly in the direction of action of the external torque.

The free gyrostat also possesses a certain degree of elasticity. It has a natural period of oscillation of its own, and its oscillations when started would continue indefinitely if undamped by friction, or otherwise. The natural period is as a rule short; if the mass of the axle and of the supporting frame of the wheel is negligible, the period is one half the time of one revolution of the wheel. If the frame is heavy, the period is increased. But unlike a perfectly elastic body, the gyrostat does not return to its original position when an external disturbing torque ceases to act on it.

The position of a gyrostat is most conveniently stated by giving its angular displacements about two perpendicular axes which are each very nearly perpendicular to the normal position of the axis of the gyrostat. In what follows it is assumed that these displacements are never large,

and the results are approximate to this extent. Any external torque acting on the gyrostat may be resolved into its components about these two axes (the third component, about the axis of the rotating wheel, not being of interest). If the reference axes be conceived to move with the frame of the gyrostat, the results may be extended to apply over large displacements of the gyrostat.

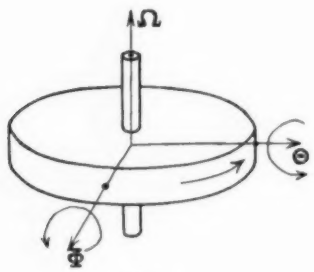

Fig. 1.

A torque $g(t)$ acting to turn the gyrostat about the Φ-axis (see Fig. 1) may be separated into two components: (1) a component which goes to accelerate the gyrostat about this axis at a rate equal to this component divided by the moment of inertia K of the wheel and mounting about this axis; (2) a component which causes precession about the Θ-axis equal to this torque-component divided by $I\omega$, where I and ω are the moment of inertia and angular velocity, respectively, of the rotating wheel. Expressed as a differential equation, this torque is

$$g(t) = K\frac{d^2\phi}{dt^2} + I\omega\frac{d\theta}{dt}. \tag{1}$$

Similarly

$$f(t) = J\frac{d^2\theta}{dt^2} - I\omega\frac{d\phi}{dt}. \tag{2}$$

Here $f(t)$ and J are respectively the torque about the Θ-axis, and the moment of inertia of the wheel and frame about the Θ-axis. The difference in sign is due to the positions of the axes relative to each other.

Differentiating (1) and (2), and combining, we have

$$JK\frac{d^3\phi}{dt^3} + I^2\omega^2\frac{d\phi}{dt} + I\omega f(t) - Jg'(t) = 0 \tag{3}$$

and

$$JK\frac{d^3\theta}{dt^3} + I^2\omega^2\frac{d\theta}{dt} - I\omega g(t) - Kf'(t) = 0. \tag{4}$$

Equation (3) will be solved in series by the parametric method devised by Professor P. A. Lambert.[1] In place of (3) we may write

$$\frac{d^3\phi}{dt^3} = -\frac{I^2\omega^2}{JK}\frac{d\phi}{dt}\cdot m - \frac{I\omega}{JK}f(t) + \frac{1}{K}g'(t), \tag{5}$$

[1] Annals of Mathematics, July, 1910.

and assume

$$\phi = \phi_0 + \phi_1 m + \phi_2 m^2 + \cdots \tag{6}$$

makes equation (5) an identity, where m is a parameter which is later placed equal to unity.

Differentiating (6) successively,

$$\frac{d\phi}{dt} = \frac{d\phi_0}{dt} + \frac{d\phi_1}{dt} \cdot m + \frac{d\phi_2}{dt} \cdot m^2 + \cdots, \tag{7}$$

$$\frac{d^3\phi}{dt^3} = \frac{d^3\phi_0}{dt^3} + \frac{d^3\phi_1}{dt^3} \cdot m + \frac{d^3\phi_2}{dt^3} \cdot m^2 + \cdots. \tag{8}$$

Inserting these values of the derivatives in (5), equating like powers of m, and solving the resulting equations in order, we readily obtain:

$$\begin{aligned}
\phi_0 &= \frac{a't^2}{\lfloor 2} + b't + c' - \frac{I\omega}{JK} f^{-3}(t) + \frac{1}{K} g^{-2}(t), \\
\phi_1 &= -\frac{I^2\omega^2}{JK} \frac{a't^4}{\underline{4}} - \frac{I^2\omega^2}{JK} \frac{b't^3}{\lfloor 3} + \frac{I^3\omega^3}{J^2K^2} f^{-5}(t) - \frac{I^2\omega^2}{JK^2} g^{-4}(t), \\
\phi_2 &= \frac{I^4\omega^4}{J^2K^2} \frac{a't^6}{\lfloor 6} + \frac{I^4\omega^4}{J^2K^2} \frac{b't^5}{\lfloor 5} - \frac{I^5\omega^5}{J^3K^3} f^{-7}(t) + \frac{I^4\omega^4}{J^2K^3} g^{-6}(t), \\
\phi^3 &= \text{etc.}
\end{aligned} \tag{9}$$

Here a', b', and c' are arbitrary constants, to be determined by the initial conditions; $f^{-1}(t)$ is written as a symbol for the definite integral $\int_0^t f(t)dt$; $f^{-2}(t)$ is a symbol for $\int_0^t \int_0^t f(t)dt^2$; etc.

Inserting equations (9) in (6), putting $m = 1$, and combining the a' and b' terms into cosine and sine series respectively, we obtain the general solutions,

$$\begin{aligned}
\phi = a \cos \frac{I\omega t}{\sqrt{JK}} + b \sin \frac{I\omega t}{\sqrt{JK}} + c - \frac{I\omega}{JK} \left[f^{-3}(t) - \frac{I^2\omega^2}{JK} f^{-5}(t) + \cdots \right] \\
+ \frac{1}{K} \left[g^{-2}(t) - \frac{I^2\omega^2}{JK} g^{-4}(t) + \cdots \right]
\end{aligned} \tag{10}$$

and from (1)

$$\begin{aligned}
\theta &= \int \left[\frac{g(t)}{I\omega} - \frac{K}{I\omega} \frac{d^2\phi}{dt^2} \right] dt \\
&= a \sqrt{\frac{K}{J}} \sin \frac{I\omega t}{\sqrt{JK}} - b \sqrt{\frac{K}{J}} \cos \frac{I\omega t}{\sqrt{JK}} + d
\end{aligned}$$

$$+\frac{1}{J}\left[f^{-2}(t)-\frac{I^2\omega^2}{JK}f^{-4}(t)+\cdots\right]$$
$$+\frac{I\omega}{JK}\left[g^{-3}(t)-\frac{I^2\omega^2}{JK}g^{-5}(t)+\cdots\right]. \quad (11)$$

Another particular solution of the general equations may be obtained by writing in place of equation (5),

$$\frac{d\phi}{dt}=-\frac{JK}{I^2\omega^2}\frac{d^3\phi}{dt^3}\cdot m-\frac{1}{I\omega}f(t)+\frac{J}{I^2\omega^2}g'(t), \quad (12)$$

and by a precisely similar method of solution are obtained:

$$\phi=a\cos\frac{I\omega t}{\sqrt{JK}}+b\sin\frac{I\omega t}{\sqrt{JK}}+c$$
$$-\frac{1}{I\omega}\left[f^{-1}(t)-\frac{JK}{I^2\omega^2}f'(t)+\frac{J^2K^2}{I^4\omega^4}f'''(t)-\cdots\right]$$
$$+\frac{J}{I^2\omega^2}\left[g(t)-\frac{JK}{I^2\omega^2}g''(t)+\cdots\right] \quad (13)$$

and

$$\theta=a\sqrt{\frac{K}{J}}\sin\frac{I\omega t}{\sqrt{JK}}-b\sqrt{\frac{K}{J}}\cos\frac{I\omega t}{JK}+d+\frac{K}{I^2\omega^2}\left[f(t)-\frac{JK}{I^2\omega^2}f''(t)+\cdots\right]$$
$$+\frac{1}{I\omega}\left[g^{-1}(t)-\frac{JK}{I^2\omega^2}g'(t)+\frac{J^2K^2}{I^4\omega^4}g'''(t)\cdots\right] \quad (14)$$

in which the complementary solutions, represented by the first three terms of the right-hand members of (10) and (11), have been included.

Either set of solutions (10) and (11), or (13) and (14), holds when $f(t)$ and $g(t)$ are analytical functions of t. If the functions $f(t)$ and $g(t)$ are arbitrary functions, or are expressed as a Fourier's series, the first set alone holds; provided in any case that the resultant series terminates or is convergent.

Placing $t=0$ in (10) and (11), we have

$$\phi_0=a+c,$$
$$\theta_0=-\sqrt{\frac{K}{J}}\cdot b+d \quad (15)$$

and by differentiation,

$$\left(\frac{d\phi}{dt}\right)_0=\frac{I\omega b}{\sqrt{JK}},$$
$$\left(\frac{d\theta}{dt}\right)_0=\frac{I\omega a}{J},$$

from which a, b, c, and d are determined. Similarly, from (13) and (14),

$$\begin{aligned}
\phi_0 &= a + c + \frac{JK}{I^3\omega^3}\left[f'(0) - \frac{JK}{I^2\omega^2}f'''(0) + \cdots\right] \\
&\qquad + \frac{J}{I^2\omega^2}\left[g(0) - \frac{JK}{I^2\omega^2}g''(0) + \cdots\right], \\
\theta_0 &= -\sqrt{\frac{K}{J}}\cdot b + d + \frac{K}{I^2\omega^2}\left[f(0) - \frac{JK}{I^2\omega^2}f''(0) + \cdots\right] \\
&\qquad - \frac{JK}{I^3\omega^3}\left[g'(0) - \frac{JK}{I^2\omega^2}g'''(0) + \cdots\right], \\
\left(\frac{d\phi}{dt}\right)_0 &= \frac{I\omega b}{\sqrt{JK}} - \frac{1}{I\omega}\left[f(0) - \frac{JK}{I^2\omega^2}f''(0) + \cdots\right] \\
&\qquad + \frac{J}{I^2\omega^2}\left[g'(0) - \frac{JK}{I^2\omega^2}g'''(0) + \cdots\right], \\
\left(\frac{d\theta}{dt}\right)_0 &= \frac{I\omega a}{\sqrt{JK}} + \frac{K}{I^2\omega^2}\left[f'(0) - \frac{JK}{I^2\omega^2}f'''(0) + \cdots\right] \\
&\qquad + \frac{1}{I\omega}\left[g(0) - \frac{JK}{I^2\omega^2}g''(0) + \cdots\right].
\end{aligned} \tag{16}$$

As an example, let a torque C be applied about the Φ-axis to a stationary gyrostat. Then

$$\phi_0 = 0; \quad \theta_0 = 0; \quad \left(\frac{d\phi}{dt}\right)_0 = 0; \quad \left(\frac{d\theta}{dt}\right)_0 = 0; \quad f(t) = 0; \quad g(t) = C.$$

Solving by means of equations (13), (14), and (16), we find,

$$\begin{aligned}
\phi &= -\frac{JC}{I^2\omega^2}\cos\frac{I\omega t}{\sqrt{JK}} + \frac{JC}{I^2\omega^2}, \\
\theta &= -\frac{\sqrt{JK}C}{I^2\omega^2}\sin\frac{I\omega t}{\sqrt{JK}} + \frac{Ct}{I\omega},
\end{aligned} \tag{17}$$

which when J is put equal to K become the results obtained by Professor Franklin.

If $g(t)$ is placed equal to kt, we readily obtain from (13), (14), and (16) the results, for a symmetrical gyrostat ($J = K$),

$$\begin{aligned}
\phi &= -\frac{J^2k}{I^3\omega^3}\sin\frac{I\omega t}{J} + \frac{Jkt}{I^2\omega^2}, \\
\theta &= \frac{J^2k}{I^3\omega^3}\cos\frac{I\omega t}{J} - \frac{J^2k}{I^3\omega^3} + \frac{kt^2}{2I\omega},
\end{aligned} \tag{18}$$

which are those given by Mr. Newkirk.

If the applied torque about the Φ-axis is periodic, we may put

$$g(t) = C \sin \alpha t.$$

Assuming as before that the gyrostat is stationary at $t = 0$, the solution, obtained from (13), (14), and (16), is

$$\begin{aligned}\phi &= \left[\frac{JC}{I^2\omega^2}\sin \alpha t - \frac{J^2\alpha C}{I^3\omega^3}\sin\frac{I\omega t}{J}\right]\left[1 + \frac{J^2\alpha^2}{I^2\omega^2} + \frac{J^4\alpha^4}{I^4\omega^4} + \cdots\right] \\ &= \frac{C}{I\omega\left(\frac{I^2\omega^2}{J^2} - \alpha^2\right)} \cdot \left[\frac{I\omega}{J}\sin \alpha t - \alpha \sin\frac{I\omega t}{J}\right], \\ \theta &= \frac{C}{I\omega\alpha\left(\frac{I^2\omega^2}{J^2} - \alpha^2\right)} \cdot \left[\frac{I^2\omega^2}{J^2}(1 - \cos \alpha t) - \alpha^2\left(1 - \cos\frac{I\omega t}{J}\right)\right].\end{aligned} \tag{19}$$

If α is put equal to $I\omega/J$ the two periods are equal, and equations (19) take the form 0/0. This may be evaluated by the rules of differential calculus, and there results,

$$\begin{aligned}\phi &= -\frac{Ct}{2I\omega}\cos\frac{I\omega t}{J} + \frac{CJ}{2I^2\omega^2}\sin\frac{I\omega t}{J}, \\ \theta &= -\frac{Ct}{2I\omega}\sin\frac{I\omega t}{J} + \frac{CJ}{I^2\omega^2}\left(1 - \cos\frac{I\omega t}{J}\right).\end{aligned} \tag{20}$$

These equations show that the amplitude of the resulting oscillations steadily increases with the time.

As a final example, consider the effect of a sudden impulse to a stationary gyrostat. Let the torque C act about the Φ-axis for the time T, T being very short. We shall use equations (10), (11), and (15) in the solution. Then $f(t) = 0$ for all values of t; $g(t) = C$ when $t < T$, and $g(t) = 0$ when $t > T$. $g^{-1}(t) = Ct$ when $t < T$, and $g^{-1}(t) = CT$ when $t > T$.

$$g^{-2}(t) = \frac{Ct^2}{2}$$

when $t < T$, and

$$g^{-2}(t) = \frac{CT^2}{2} + CT(t - T)$$

when $t > T$. Since T is very small we may without sensible error neglect all powers of T higher than the first, and $g^{-2}(t)$ becomes CTt. Similarly,

$$g^{-3}(t) = \frac{CTt^2}{\underline{2}}, \text{ etc.}$$

From equations (15) we find $a = 0 = b = c = d$. Then by (10) and (11) we have by combining the resultant quantities into sine and cosine series respectively,

$$\phi = \frac{CT}{I\omega} \sin \frac{I\omega t}{J}, \qquad \theta = \frac{CT}{I\omega}\left[1 - \cos \frac{I\omega t}{J}\right], \tag{21}$$

which give the position of the gyrostat at any time t after a sudden impact CT.

ON THE NATURE OF THE VOLTA EFFECT.

BY FERNANDO SANFORD.

THE fundamental question as to the nature of the Volta effect seems to be whether the different metals when in electrical contact with the earth or with the inside of the same hollow conductor are still at different potentials relative to each other, or whether the potential difference which may be observed between two of them when they are brought very close together is due to some kind of an electrolytic polarization in the medium between them.

The theory of electrolytic polarization has generally been so stated as to require a metallic contact between the metals before an electric field is set up between them, but the discovery of the "approach charges" of Righi and Majorana made this part of the theory untenable. It has still been maintained, however, that the electric field which is observed between the two metals when they are brought close together is not due to an original potential difference between the metals themselves, but to a potential difference between liquid or gaseous layers which are assumed to be condensed upon the metallic surfaces.

Since in order to make this theory tenable it is necessary to assume the presence of these condensed layers on the metals before they are brought near together, it seems to be implied that there is normally a double electric layer over the surface of the metal, one charge being upon the metal and the other in the liquid or gaseous layer very near the metallic surface. This view would seem to require that the surfaces of the two different metals be brought very close together before the electric field may be observed between them.

It is possible, however, to observe the approach charges without bringing the opposed metal faces very close together. One way in which this may be accomplished is by lowering an insulated conductor of one metal, as zinc, into an insulated hollow conductor of another metal, as copper. When this is done, an induced charge may be taken from the outer surface of the hollow conductor. The phenomenon is easily shown by connecting one pair of quadrants of an electrometer or the leaf of a Wilson gold-leaf electroscope to the hollow metal conductor, which is first grounded and then insulated. A ball or other convenient mass of the other metal is suspended by an insulating thread, and is first grounded

and then insulated and lowered into the hollow conductor, but is not allowed to touch it. The deflection of the electrometer needle or the gold leaf shows the induced charge upon the outer surface of the hollow conductor. If the hollow conductor be now discharged to earth and the inside conductor be withdrawn, the hollow conductor is again charged, but with the opposite sign.

It seems impossible to explain this phenomenon by any assumption of a double electric layer over the surface of the inside conductor, since such a layer must consist of equal positive and negative charges, and no number of equal positive and negative charges inside a hollow conductor could induce a charge on its outer surface.

But a still more crucial experiment may be performed with the hollow conductor. On any electrolytic theory of the Volta effect, the transference of electricity between the two metal surfaces when they are brought into contact must depend upon the nature of the opposed metallic surfaces. Accordingly, if a sphere of one metal be brought into contact with the inside of a hollow conductor of another metal, the charge which the sphere will take from the hollow conductor will depend upon the metal of the *inner* surface of the hollow conductor. If, on the other hand, the different metals when in contact with the earth are at different potentials relative to each other (since the potential of the inside of a hollow conductor must be the same as the outside) the potential difference between the inclosed sphere and the outside hollow conductor will depend upon the nature of the metal on the outside of the hollow conductor. The charge which the sphere will take when touched to the inside of the hollow conductor will, from this point of view, depend upon the metal of the *outer* surface of the hollow conductor.

It is not difficult to decide between these hypotheses. I have done so in the following manner: A gold-leaf electroscope of the C. T. R. Wilson pattern was mounted upon a stand about a foot above the supporting pier. Its outer case was connected by a copper wire to the water system of the laboratory. Its inducing plate was connected to one pole of a battery of 100 dry cells, the other pole of which was connected to earth. Its sensibility was such that one dry cell between the gold leaf and earth gave a deflection of about 14 scale divisions as seen in the reading microscope.

A polished zinc ball about 5 centimeters in diameter was suspended by a silk cord which passed over a fixed pulley about two feet above the pier, so that the ball could be raised or lowered by the cord without bringing the hand near the system. The suspending cord was about one foot from the electroscope, and a fine copper wire which was connected

to the gold leaf was suspended horizontally on an insulating support so that its end would touch the zinc ball when it was at the proper height.

Two beakers of about 750 c.c. capacity (10 cm. high and 10 cm. across the top), one of copper and one of aluminum, were used as hollow conductors. When in use, a beaker was placed upon the wooden top of the pier between three small brass screws, which served to fix its position, so that the zinc ball when lowered into it would touch the center of the bottom. A copper wire was laid between the screws so that the bottom of the beaker would rest upon it, and one end of this wire was soldered to a waterpipe in the laboratory. This insured that the beaker should be perfectly grounded during the experiment.

The zinc ball was then lowered into the beaker by means of the silk cord and pulley until it touched the bottom of the beaker, and was then drawn up until it touched the wire from the electroscope. As soon as the ball was outside the beaker it had a free charge which it shared with the gold leaf of the electroscope when the two were in contact. On account of the small capacity of the gold leaf and connecting wire, a very few repetitions of this process would bring the gold leaf to practically the potential of the zinc ball when it left the beaker. A series of readings were then made by successively lowering the ball to the bottom of the beaker and raising it until it touched the charging wire of the electroscope, then lowering the ball and reading the electroscope.

These readings were not constant, but varied among themselves by several scale divisions. This was probably due in part to the uncertain contact which the charging wire of the electroscope made with the zinc ball, but no special attempt was made to get rid of the various disturbing causes which affect electrostatic experiments in all laboratories. The means of several series of ten readings were found to be nearly enough constant for the purposes of this investigation, and were accordingly used.

Copper Beaker.	Aluminum Beaker.	Copper Beaker.	Aluminum Beaker.
16	19	27	23
16	29	25	28
16	35	16	31
18	37	15	32
11	40	17	34
11	42	15	40
11	44	10	37
10	39	10	40
10	35	10	44
10	32	11	40
Mean....12.9	35.2	15.6	34.9

In determining the difference between the charges induced by the two beakers, the method adopted was to make ten successive readings with one beaker, then, while the zinc ball was raised the other beaker was substituted and ten readings were made with it, when the beakers were again changed. The following is a sample series made by using the zinc ball with the aluminum and copper beakers. The series was made without grounding the electroscope or making any change in the apparatus except to substitute one beaker for the other at the end of ten readings.

The mean of the two sets from the copper beaker and the two sets from the aluminum beaker differed by 20.8 scale divisions. The electroscope at that time gave about 14 scale divisions for one dry cell, or approximately 10 scale divisions for one volt. Of course, this difference of 20.8 scale divisions does not represent the true Volta potential difference between the aluminum and the copper beakers, as whenever the beakers were exchanged several charges of the ball were required to bring the electroscope charge up to a constant value.

Two circular discs of tin foil ten centimeters in diameter which had been cut from the same sheet were then pressed down into the beakers, so that they covered the bottoms and were turned up one centimeter around the edges. Series of readings were then taken with the two beakers as before. In this case, the zinc ball touched only the center of the tin-foil disc, and if its charge were due to an electrolytic action between it and the tin foil the charge should have been the same in both beakers. The following series of readings were taken from the two beakers with the tin foil in the bottom:

Tin Foil in Aluminum.	Tin Foil in Copper.	Tin Foil in Aluminum.	Tin Foil in Copper.
44	31	34	20
45	28	31	29
47	22	31	23
45	22	35	20
39	17	33	25
42	15	37	18
41	18	43	15
41	22	45	14
41	25	47	13
38	20	49	13
Mean....42.3	22	38.5	19

The means of these four series give a difference between the charges taken from the tin foil in the aluminum beaker and those taken from a disc of the same tin foil in the copper beaker of 20 scale divisions, virtually the same difference which was observed between the two beakers them-

selves. This experiment was repeated several times with similar results. Sometimes the difference between the charges taken from the tin foil was a little greater than between those taken from the naked beaker, and sometimes a little less. The tin-foil discs were interchanged without affecting the result.

The beakers were then inverted and two similar discs of tin foil were placed upon the *outside* of the bottoms. The beakers were then placed in their former positions inverted, and the zinc ball was then charged from the center of the tin-foil discs on the outside of the beakers. The following series of readings were made in this way. They differed from the other series only in that 15 readings were taken from each beaker and the first five rejected. This was because of the fact that the ball would take a much smaller charge from the outside of the beaker, and if there were a difference in the potentials of the two tin-foil discs it would take more charges to show it. The result would, however, have been scarcely changed by taking the mean of the 15.

Tin Foil on Copper.	Tin Foil on Aluminum.
35	38
35	40
37	42
34	36
36	36
39	36
35	32
31	34
35	35
36	36
Mean....35.3	36.5

The difference of 1.2 scale divisions between the means of these two series is not greater than the probable error of the experiment.

Beakers Covered with Tin Foil.		Beakers Uncovered.	
Aluminum.	Copper.	Copper.	Aluminum.
51	40	26	46
52	46	23	45
50	44	26	51
55	39	28	50
54	45	26	52
57	43	25	55
52	45	32	56
51	39	31	53
46	46	31	49
41	48	29	54
Mean....50.9	43.4	27.7	51.1

The experiment was then varied by wrapping the outside of the beakers with tin foil, leaving an opening in the top of about 7 cm. in diameter. The beakers were placed upright and the copper ball charged from the inside. Ten readings were taken from each beaker, as before. The tin foil was then removed and ten readings taken again with nothing changed but the tin foil. The following series is the result.

In this experiment, though the beakers were not wholly enclosed in the tin foil the charges taken from their interior surfaces differed by only one third as much as when the tin foil was removed.

These experiments were in part repeated with a gilt ball instead of the zinc ball, but without changing the result. As said before, no claim is made that the actual potential difference between the beakers was obtained with any high degree of accuracy in any case. The different series of readings are, however, fairly comparable, and only one conclusion seems possible from them. Apparently the Volta effect is not due to any electrolytic action between the opposed metals, but different metals when in contact with the earth or with the inside of the same hollow conductor may be at different potentials relative to each other.

STANFORD UNIVERSITY,
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